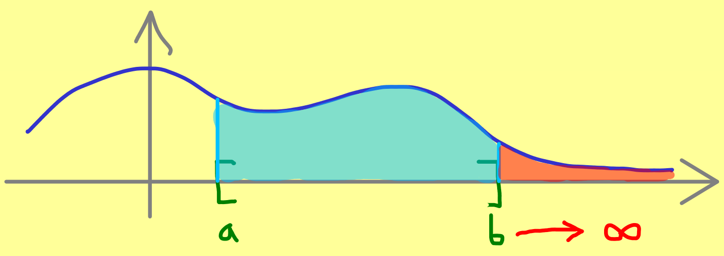
The Bright Side of Mathematics

Real Analysis – Part 60

$f : \mathbb{R} \to \mathbb{R}$ continuous

$\int_a^b f(x)\,dx$ well-defined for $a, b \in \mathbb{R}$

$\int_a^\infty f(x)\,dx = ?$

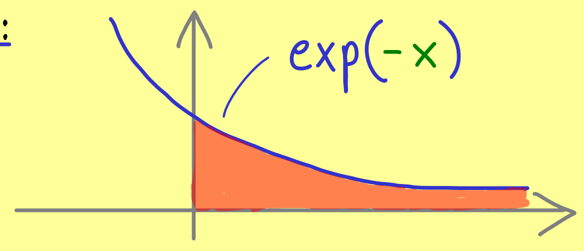


Definition: $f : [a, \infty) \to \mathbb{R}$ be a function with the property:

$$f|_{[a,b]} \in \mathcal{R}([a,b]) \quad \text{for all } b \geq a$$

If $\lim\limits_{b \to \infty} \int_a^b f(x)\,dx$ exists, we write $\int_a^\infty f(x)\,dx$ for this limit and we say the integral converges.

Example:



$$\int_0^\infty \exp(-x)\,dx = \lim_{b \to \infty} \int_0^b \exp(-x)\,dx$$

$$= \lim_{b \to \infty} \left(-\exp(-x) \Big|_0^b \right)$$

$$= \lim_{b \to \infty} \left(-\exp(-b) + 1 \right) = 1$$

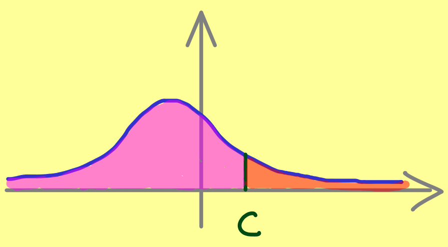
Similar definition for: $\int_{-\infty}^b f(x)\,dx$

Definition: $f : \mathbb{R} \to \mathbb{R}$ be a function with the property:

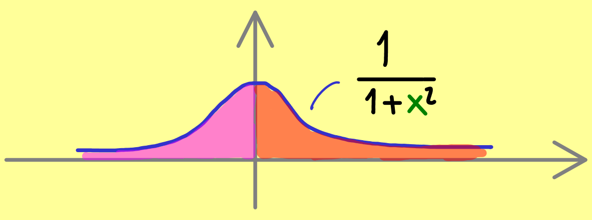
$$f|_{[a,b]} \in \mathcal{R}([a,b]) \quad \text{for all } a, b \in \mathbb{R} \quad (a < b)$$

If there is a $c \in \mathbb{R}$ such that $\int_{-\infty}^c f(x)\,dx$ and $\int_c^\infty f(x)\,dx$ converge,

$$\int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx$$



Example:



$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx = \int_{-\infty}^0 \frac{1}{1+x^2}\,dx + \int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{a \to -\infty} \int_a^0 \frac{1}{1+x^2}\,dx + \lim_{b \to \infty} \int_0^b \frac{1}{1+x^2}\,dx$$

$$= \lim_{a \to -\infty} \arctan(x)\Big|_a^0 + \lim_{b \to \infty} \arctan(x)\Big|_0^b$$

$$= \lim_{b \to \infty} \arctan(b) - \lim_{a \to -\infty} \arctan(a)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$