BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Real Analysis – Part 59

$$\int_3^5 \frac{1}{x(x+1)}\,dx = ? \qquad \text{antiderivative?}$$

partial fraction decomposition

$$\frac{1}{x(x+1)} \overset{\checkmark}{=} \frac{1}{x} - \frac{1}{x+1} \qquad \left(= \frac{1\cdot(x+1)}{x\cdot(x+1)} - \frac{x\cdot 1}{x\cdot(x+1)} \right)$$

antiderivative: $\displaystyle\int \frac{1}{x(x+1)}\,dx = \int \left(\frac{1}{x} - \frac{1}{x+1}\right) dx = \int \frac{1}{x}\,dx - \int \frac{1}{x+1}\,dx$

$$= \log(|x|) - \log(|x+1|) \quad + \text{ constant}$$

Partial fraction decomposition: Let f be a rational function

$$f(x) = \frac{p(x)}{q(x)} \quad \text{with} \quad \deg(p) < \deg(q) =: n$$

We need the zeros of q:

(1) n different real zeros: $x_1, x_2, \dots, x_n$

$$\frac{p(x)}{q(x)} = \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \dots + \frac{A_n}{x-x_n} \qquad \text{Find } A_1, \dots, A_n\,!$$

(2) k different real zeros: $x_1, x_2, \dots, x_k$ with multiplicities $\alpha_1, \dots, \alpha_k$

$$\sum_{j=1}^{k} \alpha_j = n$$

$$\frac{p(x)}{q(x)} = \frac{A_1^{(1)}}{x-x_1} + \frac{A_1^{(2)}}{(x-x_1)^2} + \dots + \frac{A_1^{(\alpha_1)}}{(x-x_1)^{\alpha_1}} + \frac{A_2^{(1)}}{x-x_2} + \frac{A_2^{(2)}}{(x-x_2)^2} + \dots$$

(3) q has complex zeros: calculate as in (1) and (2) with

$$x_1, x_2, \dots, x_k \in \mathbb{C}, \quad A_1^{(1)}, \dots, A_k^{(\alpha_k)} \in \mathbb{C}$$

Example: $f(x) = \dfrac{1}{x^2(x-1)}$, zeros of the denominator: $x_1 = 0$, $x_2 = 1$; $\alpha_1 = 2$, $\alpha_2 = 1$

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} \qquad \Big|\cdot x^2(x-1)$$

$$\Rightarrow \quad 1 = A\cdot x(x-1) + B\cdot(x-1) + C\cdot x^2$$

$$\Rightarrow \quad 1 = x^2\cdot(A+C) + x\cdot(-A+B) + 1\cdot(-B)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right)$$

$$\overset{\text{II}+\text{I}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{array}\right) \overset{\text{III}+\text{II}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\Rightarrow \quad C = 1, \quad B = -1, \quad A = -1$$

$$\Rightarrow \quad \int \frac{1}{x^2(x-1)}\,dx = \int \frac{-1}{x}\,dx + \int \frac{-1}{x^2}\,dx + \int \frac{1}{(x-1)}\,dx$$

$$= -\log(|x|) + \frac{1}{x} + \log(|x-1|) \quad + \text{ constant}$$