

The Bright Side of Mathematics



Real Analysis - Part 50

Riemann integral for step function:

$$\int_a^b \phi(x) dx$$

map: $S([a, b]) \rightarrow \mathbb{R}$

$$\phi \mapsto \int_a^b \phi(x) dx$$

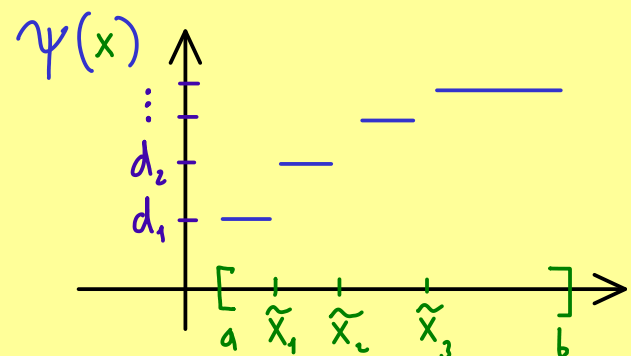
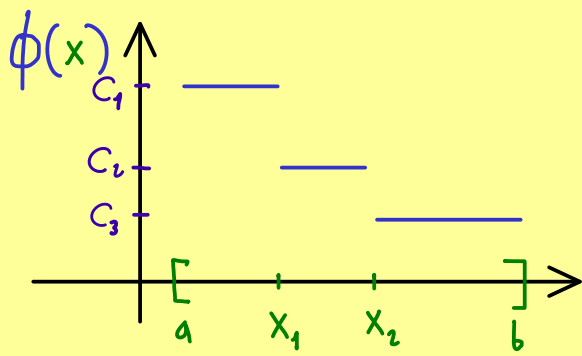
is linear and monotonic

Proposition: (1) For $\lambda \in \mathbb{R}$: $\int_a^b \lambda \phi(x) dx = \lambda \cdot \int_a^b \phi(x) dx$ (homogeneous)

(2) For $\phi, \psi \in S([a, b])$: $\int_a^b (\phi + \psi)(x) dx = \int_a^b \phi(x) dx + \int_a^b \psi(x) dx$ (additive)

(3) For $\phi, \psi \in S([a, b])$: $\phi \leq \psi \Rightarrow \int_a^b \phi(x) dx \leq \int_a^b \psi(x) dx$ (monotonic)

Proof: (2)



$$\mathcal{P}_1: a = x_0 < x_1 < \dots < x_n = b$$

$$\mathcal{P}_2: a = \tilde{x}_0 < \tilde{x}_1 < \dots < \tilde{x}_m = b$$

Define: $\mathcal{P}_3 = \mathcal{P}_1 \cup \mathcal{P}_2$: $a = \tilde{\tilde{x}}_0 < \tilde{\tilde{x}}_1 < \dots < \tilde{\tilde{x}}_N = b$

$$\begin{aligned} \int_a^b \phi(x) dx + \int_a^b \psi(x) dx &= \sum_{j=1}^N c_j \cdot (\tilde{\tilde{x}}_j - \tilde{\tilde{x}}_{j-1}) + \sum_{j=1}^N d_j \cdot (\tilde{\tilde{x}}_j - \tilde{\tilde{x}}_{j-1}) \\ &= \sum_{j=1}^N (c_j + d_j) (\tilde{\tilde{x}}_j - \tilde{\tilde{x}}_{j-1}) = \int_a^b (\phi + \psi)(x) dx \end{aligned}$$