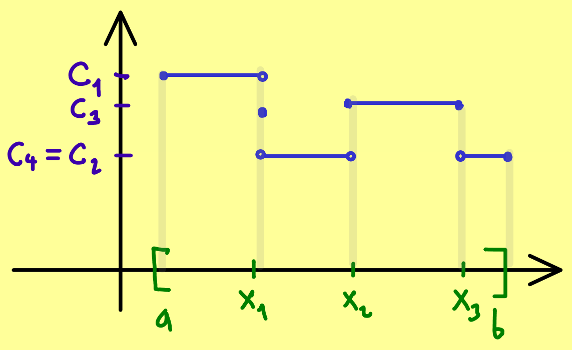
The Bright Side of
Mathematics

Real Analysis – Part 49

$\phi : [a,b] \longrightarrow \mathbb{R}$ is called a step function if it is piecewisely constant:



there is a partition of $[a,b]$, $\{x_0, x_1, \dots, x_n\}$, and there are numbers $c_1, \dots, c_n \in \mathbb{R}$ such that

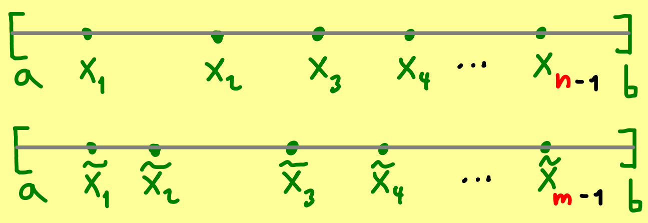
$$\phi|_{(x_{j-1},\, x_j)} = c_j \qquad \text{for all } j \in \{1, \dots, n\}$$

Proposition: $\displaystyle\int_a^b \phi(x)\, dx := \sum_{j=1}^{n} c_j \cdot (x_j - x_{j-1})$ is well-defined.

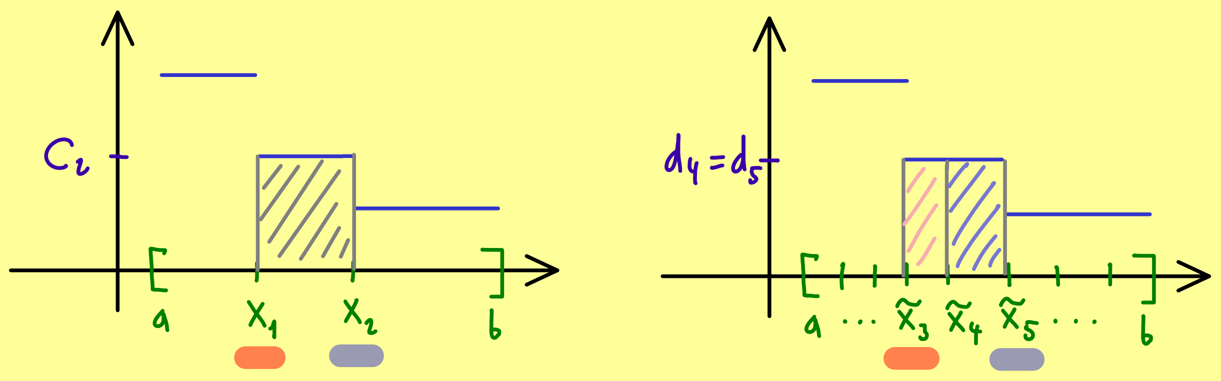
Proof:

$P_1: \quad a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$

$P_2: \quad a = \tilde{x}_0 < \tilde{x}_1 < \tilde{x}_2 < \cdots < \tilde{x}_{m-1} < \tilde{x}_m = b$



with $\phi|_{(x_{j-1},\, x_j)} = c_j$, $\phi|_{(\tilde{x}_{j-1},\, \tilde{x}_j)} = d_j$



First case: $P_2 \supset P_1$ (partition 2 is finer than partition 1)

For example: $x_1 = \tilde{x}_3 < \tilde{x}_4 < \tilde{x}_5 = x_2$, $c_2 = d_4 = d_5$

$$d_4 \cdot (\tilde{x}_4 - \tilde{x}_3) + d_5 \cdot (\tilde{x}_5 - \tilde{x}_4) = c_2 \cdot (\tilde{x}_4 - \underbrace{\tilde{x}_3}_{x_1} + \underbrace{\tilde{x}_5}_{x_2} - \tilde{x}_4) = c_2 \cdot (x_2 - x_1)$$

$$\sum_{j=1}^{n} c_j \cdot (x_j - x_{j-1}) = \sum_{j=1}^{m} d_j \cdot (\tilde{x}_j - \tilde{x}_{j-1})$$

Second case: $P_2 \not\supset P_1$ and $P_1 \not\supset P_2$: $\quad P_3 := P_1 \cup P_2$

$$\Rightarrow \quad P_3 \supset P_1 \text{ and } P_3 \supset P_2$$

$$\Rightarrow \quad \sum_{P_1} = \sum_{P_3} \text{ and } \sum_{P_2} = \sum_{P_3} \Rightarrow \sum_{P_1} = \sum_{P_2}$$