The Bright Side of Mathematics

Real Analysis – Part 47

Taylor:

$$f(x_0+h) = T_n(h) + R_n(h)$$

$$T_n(h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} \cdot h^k}_{n\text{-th order Taylor polynomial}} \qquad R_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot h^{n+1}$$

ξ between x_0 and x_0+h

Proof:

$$F_{n,h}(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} \cdot (h + x_0 - t)^k$$

Note: $F_{n,h}(x_0) = T_n(h)$

$F_{n,h}(x_0+h) = f(x_0+h)$

$$g_{n,h}(t) := (h + x_0 - t)^{n+1} , \quad g'_{n,h}(t) = -(n+1)\cdot(h + x_0 - t)^n$$

Generalised mean value theorem:

$$\frac{F_{n,h}(x_0+h) - F_{n,h}(x_0)}{g_{n,h}(x_0+h) - g_{n,h}(x_0)} = \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)}$$

ξ between x_0 and x_0+h

$$f(x_0+h) - T_n(h) = \Big(\underbrace{g_{n,h}(x_0+h)}_{0} - \underbrace{g_{n,h}(x_0)}_{h^{n+1}}\Big) \frac{F'_{n,h}(\xi)}{g'_{n,h}(\xi)} = \frac{h^{n+1} \cdot F'_{n,h}(\xi)}{(n+1)\cdot(h + x_0 - \xi)^n}$$

$$= \frac{h^{n+1} \cdot \frac{f^{(n+1)}(\xi)}{n!} \cdot (h + x_0 - \xi)^n}{(n+1)\cdot(h + x_0 - \xi)^n}$$

$$= h^{n+1} \cdot \frac{f^{(n+1)}(\xi)}{(n+1)!}$$

$$F'_{n,h}(t) = \frac{d}{dt} \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!} \cdot (h + x_0 - t)^k$$

$$= \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!} \cdot (h+x_0-t)^k - \sum_{k=1}^{n} \frac{f^{(k)}(t)}{(k-1)!} \cdot (h+x_0-t)^{k-1}$$

$$= \frac{f^{(n+1)}(t)}{n!} \cdot (h+x_0-t)^n$$