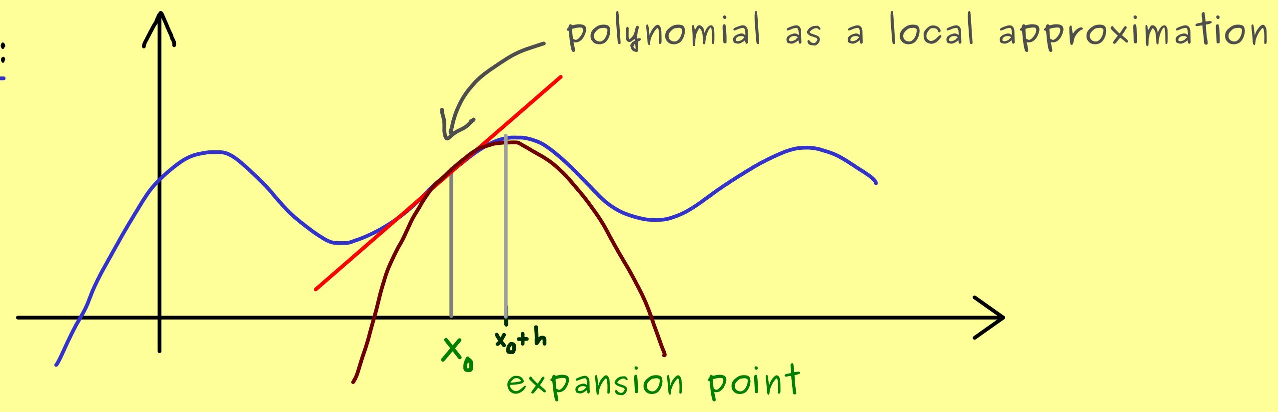
The Bright Side of Mathematics

Real Analysis – Part 45

Taylor's theorem:



Linear approximation: $(x = x_0 + h)$ $\quad f(x_0+h) = f(x_0) + f'(x_0)\cdot h + r(h)\cdot h$ with $r(h) \xrightarrow{h\to 0} 0$

Quadratic approximation: $\quad f(x_0+h) = f(x_0) + f'(x_0)\cdot h + \frac{1}{2}\cdot f''(x_0)\cdot h^2 + r(h)\cdot h^2$

with $r(h) \xrightarrow{h\to 0} 0$

Theorem: I interval, $f: I \longrightarrow \mathbb{R}$ $(n+1)$-differentiable, $x_0 \in I$.

If $h \in \mathbb{R}$ such that $x_0 + h \in I$, then:

$$f(x_0+h) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot h^k}_{\substack{n\text{-th order} \\ \text{Taylor polynomial}}} + \underbrace{R_n(h)}_{\text{remainder term}}$$

and there is ξ with $\xi \in (x_0, x_0+h)$ or $\xi \in (x_0+h, x_0)$

such that $R_n(h) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot h^{n+1}$

One often writes: $f(x_0+h) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot h^k + \mathcal{O}(h^{n+1})$ (Landau symbol)

Or with $(x = x_0 + h)$: $\quad f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\cdot (x-x_0)^k + \mathcal{O}\left((x-x_0)^{n+1}\right)$