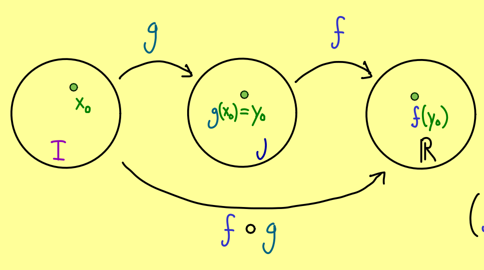
The Bright Side of Mathematics

Real Analysis – Part 36



$$(f \circ g)(x) = f(g(x))$$

Chain rule: Let $I, J \subseteq \mathbb{R}$ be two intervals and $g: I \longrightarrow J$, $f: J \longrightarrow \mathbb{R}$.

g differentiable at x_0

f differentiable at $y_0 = g(x_0)$

$\Longrightarrow$ $f \circ g$ differentiable at x_0 and:

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0)$$

$$\left.\frac{d f(g(x))}{dx}\right|_{x_0} = \left.\frac{d f(y)}{dy}\right|_{g(x_0)} \cdot \left.\frac{d g(x)}{dx}\right|_{x_0}$$

Proof: $g(x) = g(x_0) + (x - x_0) \cdot \Delta_{g, x_0}(x)$, $\quad f(y) = f(y_0) + (y - y_0) \cdot \Delta_{f, y_0}(y)$, $y_0 = g(x_0)$

$$(f \circ g)(x) = f(\underbrace{g(x)}_{y \in J}) = f(y_0) + (g(x) - y_0) \cdot \Delta_{f, y_0}(g(x))$$

$$= f(y_0) + \left(g(x_0) + (x - x_0) \cdot \Delta_{g, x_0}(x) - y_0\right) \cdot \Delta_{f, y_0}(g(x))$$

$$= f(y_0) + (x - x_0) \cdot \underbrace{\Delta_{g, x_0}(x) \cdot \Delta_{f, y_0}(g(x))}_{\text{continuous at } x_0}$$

$$= (f \circ g)(x_0) + (x - x_0) \cdot \Delta_{f \circ g, x_0}(x)$$

$\Longrightarrow$ $f \circ g$ differentiable at x_0 with $(f \circ g)'(x_0) = g'(x_0) \cdot f'(g(x_0)) = f'(g(x_0)) \cdot g'(x_0)$