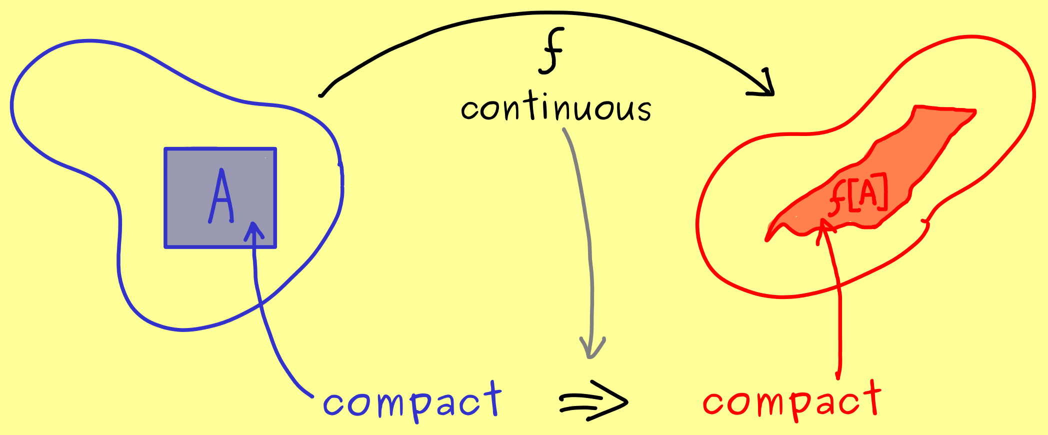
The Bright Side of Mathematics

Real Analysis – Part 30



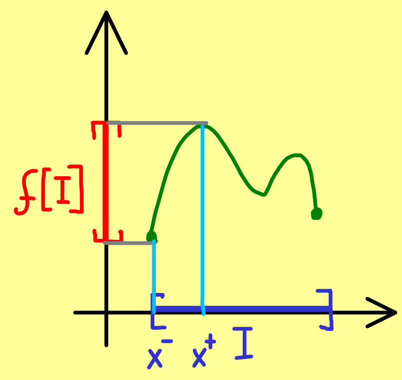
Theorem: $I \subseteq \mathbb{R}$ compact, $f: I \to \mathbb{R}$ continuous.

Then: $f[I] \subseteq \mathbb{R}$ is compact ($\overset{\text{Heine–Borel}}{=}$ bounded + closed)

and there are $x^+, x^- \in I$ with

$$f(x^+) = \sup\{ f(x) \mid x \in I \}$$

$$f(x^-) = \inf\{ f(x) \mid x \in I \}$$



Proof: Compact means: every sequence has a convergent subsequence.

Let $(y_n)_{n\in\mathbb{N}} \subseteq f[I]$ be a sequence.

For each y_n there is $x_n \in I$ with $f(x_n) = y_n$. $\Rightarrow$ New sequence $(x_n)_{n\in\mathbb{N}} \subseteq I$

$\overset{I \text{ compact}}{\Longrightarrow}$ There is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ that is convergent: $x := \lim\limits_{k\to\infty} x_{n_k} \in I$

$$\lim_{k\to\infty} y_{n_k} = \lim_{k\to\infty} f(x_{n_k}) \underset{\uparrow \, f \text{ continuous}}{=} f\left(\lim_{k\to\infty} x_{n_k}\right) = f(x) =: y$$

So $(y_{n_k})_{k\in\mathbb{N}}$ is convergent with limit $y \in f[I]$. $\Rightarrow$ $f[I]$ compact