The Bright Side of Mathematics

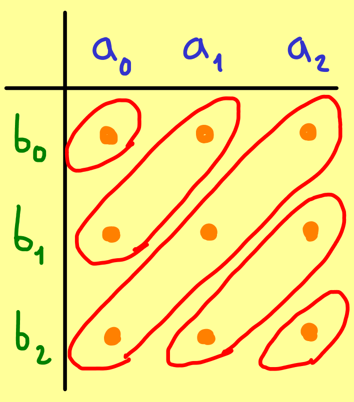
Real Analysis – Part 22

$$\sum_{k=0}^{\infty} a_k \,,\; \sum_{k=0}^{\infty} b_k \quad \xrightarrow{\text{How to multiply?}} \quad \sum_{k=0}^{\infty} c_k$$

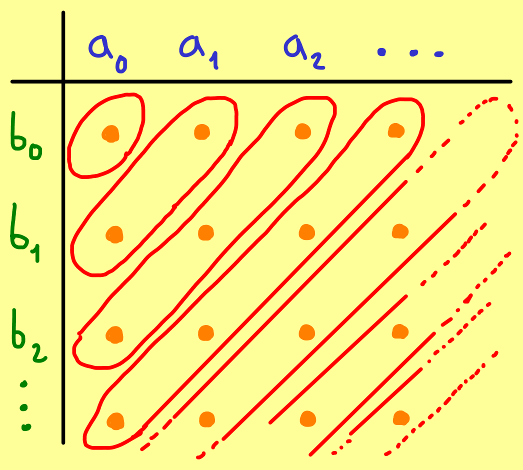
For finite sums: $(a_0 + a_1 + a_2)\cdot(b_0 + b_1 + b_2)$

$$= a_0 b_0 + a_1 b_0 + a_2 b_0 + a_0 b_1 + a_1 b_1 + a_2 b_1 + a_0 b_2 + a_1 b_2 + a_2 b_2$$

$$= \underbrace{(a_0 b_0)}_{0} + \underbrace{(a_1 b_0 + a_0 b_1)}_{1} + \underbrace{(a_2 b_0 + a_1 b_1 + a_0 b_2)}_{2} + \underbrace{(a_2 b_1 + a_1 b_2)}_{3} + \underbrace{(a_2 b_2)}_{4}$$



Cauchy product: For two series $\sum_{k=0}^{\infty} a_k$, $\sum_{k=0}^{\infty} b_k$, the series $\sum_{k=0}^{\infty} c_k$ with $c_k = \sum_{\ell=0}^{k} a_\ell \, b_{k-\ell}$ is called the Cauchy product.



Theorem: If $\sum_{k=0}^{\infty} a_k$ is absolutely convergent and $\sum_{k=0}^{\infty} b_k$ convergent, then Cauchy product $\sum_{k=0}^{\infty} c_k$ is abs. convergent and $\sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\cdot\left(\sum_{k=0}^{\infty} b_k\right)$

Example: $\exp(x) := \sum_{k=0}^{\infty} \frac{x^k}{k!}$ for $x \in \mathbb{R}$ (abs. convergent by the ratio test)

Apply Cauchy product for $\exp(x)$ and $\exp(y)$:

$$c_k = \sum_{\ell=0}^{k} \frac{x^\ell}{\ell!}\cdot\frac{y^{k-\ell}}{(k-\ell)!} = \frac{1}{k!}\sum_{\ell=0}^{k}\binom{k}{\ell} x^\ell y^{k-\ell}$$

binomial coefficient: $\binom{k}{\ell} = \frac{k!}{\ell!\,(k-\ell)!}$

$$\stackrel{\text{binomial theorem}}{=} \frac{1}{k!}(x+y)^k$$

$$\exp(x+y) = \sum_{k=0}^{\infty}\frac{1}{k!}(x+y)^k = \sum_{k=0}^{\infty} c_k = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \exp(x)\cdot\exp(y)$$

$$\Longrightarrow \quad \boxed{\exp(x+y) = \exp(x)\cdot\exp(y)}$$

fundamental multiplicative identity