The Bright Side of Mathematics

Real Analysis – Part 17

Series: $\sum_{k=1}^{\infty} a_k$ sequence of partial sums

Properties: If $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent, $\lambda \in \mathbb{R}$, then:

(a) $\sum_{k=1}^{\infty} (a_k + b_k)$ is also convergent

and the limit is: $\sum_{k=1}^{\infty} (a_k + b_k) = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty} b_k$

(b) $\sum_{k=1}^{\infty} (\lambda \cdot a_k)$ is also convergent

and the limit is: $\sum_{k=1}^{\infty} (\lambda \cdot a_k) = \lambda \cdot \sum_{k=1}^{\infty} a_k$

Cauchy criterion: $\sum_{k=1}^{\infty} a_k$ is convergent $\iff$ $\forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq m \geq N :$

$$\left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

Proof: $S_n := \sum_{k=1}^{n} a_k$. $(S_n)_{n \in \mathbb{N}}$ is convergent $\overset{\text{completeness}}{\iff}$ $(S_n)_{n \in \mathbb{N}}$ is a Cauchy sequence

$\iff \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall \tilde{n}, \tilde{m} \geq N : \; |S_{\tilde{n}} - S_{\tilde{m}}| < \varepsilon$

$\iff \forall \varepsilon > 0 \; \exists N \in \mathbb{N} \; \forall n \geq m \geq N : \; |S_n - S_{m-1}| < \varepsilon$

Example: $\sum_{k=1}^{\infty} (-1)^k$ Calculate: $\left| \sum_{k=N}^{N+2} (-1)^k \right| = \begin{cases} |1 + (-1) + 1| \\ |-1 + 1 + (-1)| \end{cases} = 1$

Important fact: $\sum_{k=1}^{\infty} a_k$ is convergent $\Rightarrow$ $(a_k)_{k \in \mathbb{N}}$ convergent with $\lim_{k \to \infty} a_k = 0$