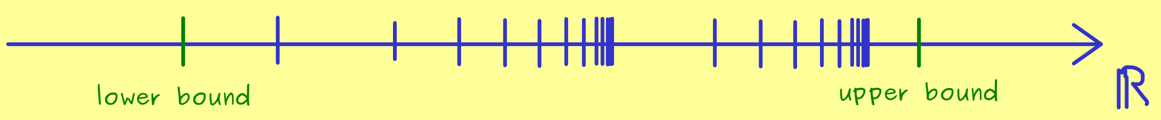
The Bright Side of
Mathematics

Real Analysis – Part 10

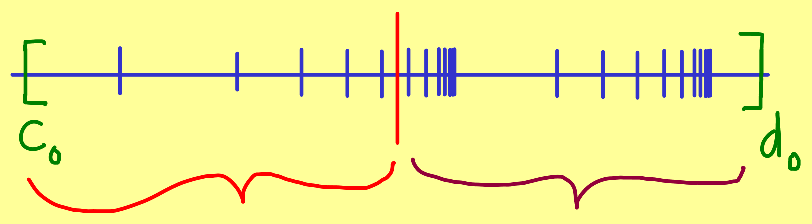
Bolzano–Weierstrass theorem

$(a_n)_{n\in\mathbb{N}}$ bounded $\implies$ $(a_n)_{n\in\mathbb{N}}$ has an accumulation value

(has a convergent subsequence)



Proof:

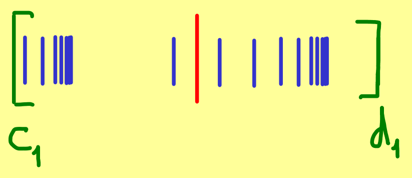


If infinitely many sequence members in it: Choose left-hand interval

Otherwise: Choose right-hand interval

repeat

New interval:



We get: $[c_0, d_0] \supset [c_1, d_1] \supset [c_2, d_2] \supset [c_3, d_3] \supset \dots$

And: $d_1 - c_1 = \frac{1}{2}(d_0 - c_0)$, $d_2 - c_2 = \frac{1}{2}(d_1 - c_1) = \frac{1}{4}(d_0 - c_0)$, ...

$$d_n - c_n = \frac{1}{2^n}(d_0 - c_0) \xrightarrow{n\to\infty} 0$$

We know: $\left.\begin{array}{l}(c_n)_{n\in\mathbb{N}} \text{ mon. increasing and bounded} \\ (d_n)_{n\in\mathbb{N}} \text{ mon. decreasing and bounded}\end{array}\right\} \implies (c_n)_{n\in\mathbb{N}}, (d_n)_{n\in\mathbb{N}}$ convergent

By limit theorems: $0 = \lim\limits_{n\to\infty}(d_n - c_n) = \lim\limits_{n\to\infty} d_n - \lim\limits_{n\to\infty} c_n$

Define a subsequence $(a_{n_k})_{k\in\mathbb{N}}$ by choosing $a_{n_k} \in [c_k, d_k]$

$$\implies c_k \leq a_{n_k} \leq d_k$$

$$\overset{\text{Sandwich theorem}}{\implies} (a_{n_k})_{k\in\mathbb{N}} \text{ is convergent}$$