



The Bright Side of Mathematics

Real Analysis - Part 8

Fact: If $(a_n)_{n \in \mathbb{N}}$ is monotonically **increasing** ($a_{n+1} \geq a_n$ for all n) and bounded from **above** (the set $\{a_n\}_{n \in \mathbb{N}}$ has an upper bound), then: $(a_n)_{n \in \mathbb{N}}$ is convergent. (Monotone convergence criterion)

Example: The sequence $(a_n)_{n \in \mathbb{N}}$ given by $a_n = \left(1 + \frac{1}{n}\right)^n$ is convergent.

Proof: (1) Monotonicity: $\frac{a_{n+1}}{a_n}$ $\left(\begin{array}{l} \leq 1 \text{ mon. decreasing} \\ \geq 1 \text{ mon. increasing} \end{array} \right)$

$$\frac{a_{n+1}}{a_n} = \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} = \left(1 + \frac{1}{n}\right) \cdot \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^{n+1}} = \left(1 + \frac{1}{n}\right) \left(\frac{\left(1 + \frac{1}{n+1}\right)^{n(n+1)}}{\left(1 + \frac{1}{n}\right)^{n(n+1)}} \right)^{n+1}$$

$$= \left(1 + \frac{1}{n}\right) \left(\frac{n(n+1) + n + 1 - 1}{n(n+1) + n + 1} \right)^{n+1} = \left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{\underbrace{n^2 + 2n + 1}_{(n+1)^2}} \right)^{n+1}$$

Bernoulli's inequality:
For $k \in \mathbb{N}$ and $x \geq -1$
 $(1+x)^k \geq 1+k \cdot x$

$$\geq \left(1 + \frac{1}{n}\right) \left(1 + \cancel{(n+1)} \cdot \left(-\frac{1}{\cancel{(n+1)^2}}\right)\right)$$

$$= \left(\frac{\cancel{n+1}}{n}\right) \cdot \left(\frac{\cancel{n}}{\cancel{n+1}}\right) = \underline{1} \quad \checkmark$$

(2) Bounded from above: $a_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^n \binom{n}{k} 1^{n-k} \left(\frac{1}{n}\right)^k$

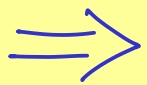
$$= \underbrace{\binom{n}{0}}_{=1} \cdot 1^n \cdot \underbrace{\left(\frac{1}{n}\right)^0}_{=1} + \underbrace{\binom{n}{1}}_n \cdot 1^{n-1} \left(\frac{1}{n}\right)^1 + \sum_{k=2}^n \binom{n}{k} \left(\frac{1}{n}\right)^k$$

$$= 1 + 1 + \sum_{k=2}^n \binom{n}{k} \left(\frac{1}{n}\right)^k \leq 2 + 1 - \frac{1}{n} \leq 3$$

We have: $\binom{n}{k} \cdot \left(\frac{1}{n}\right)^k = \frac{n!}{(n-k)! \cdot k!} \cdot \left(\frac{1}{n^k}\right) = \frac{n \cdot \underbrace{(n-1)(n-2) \dots (n-k+1)}_{\leq 1}}{n \cdot n \cdot n \dots n} \cdot \frac{1}{k!} \leq \frac{1}{k!}$

$$\leq \frac{1}{k \cdot (k-1)} = \frac{1}{k-1} - \frac{1}{k} \quad \text{and} \quad \sum_{k=2}^n \left(\frac{1}{k-1} - \frac{1}{k}\right) \stackrel{\text{telescoping}}{=} 1 - \frac{1}{n}$$

fact



The sequence $(a_n)_{n \in \mathbb{N}}$ is convergent.

Monotone convergence criterion

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n =: e \quad \text{Euler's number}$$