The Bright Side of Mathematics

Real Analysis – Part 8

Fact: If $(a_n)_{n\in\mathbb{N}}$ is monotonically increasing ($a_{n+1} \geq a_n$ for all n) and bounded from above (the set $\{a_n\}_{n\in\mathbb{N}}$ has an upper bound), then: $(a_n)_{n\in\mathbb{N}}$ is convergent.

(Monotone convergence criterion)

Example: The sequence $(a_n)_{n\in\mathbb{N}}$ given by $a_n = \left(1+\frac{1}{n}\right)^n$ is convergent.

Proof: (1) Monotonicity: $\frac{a_{n+1}}{a_n} \begin{pmatrix} \leq 1 & \text{mon. decreasing} \\ \geq 1 & \text{mon. increasing} \end{pmatrix}$

$$\frac{a_{n+1}}{a_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = \left(1+\frac{1}{n}\right)\cdot\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n+1}} = \left(1+\frac{1}{n}\right)\left(\frac{\left(1+\frac{1}{n+1}\right)^{n(n+1)}}{\left(1+\frac{1}{n}\right)^{n(n+1)}}\right)$$

$$= \left(1+\frac{1}{n}\right)\left(\frac{n(n+1)+n+1-1}{n(n+1)+n+1}\right)^{n+1} = \left(1+\frac{1}{n}\right)\left(1 - \frac{1}{\underbrace{n^2+2n+1}_{(n+1)^2}}\right)^{n+1}$$

Bernoulli's inequality:
For $k\in\mathbb{N}$ and $x \geq -1$
$$(1+x)^k \geq 1 + k\cdot x$$

$$\geq \left(1+\frac{1}{n}\right)\left(1 + (n+1)\cdot\left(-\frac{1}{(n+1)^2}\right)\right)$$

$$= \left(\frac{n+1}{n}\right)\cdot\left(\frac{n}{n+1}\right) = 1 \quad \checkmark$$

(2) Bounded from above: $a_n = \left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}1^{n-k}\left(\frac{1}{n}\right)^k$

$$= \underbrace{\binom{n}{0}}_{=1}\cdot 1^n\cdot\underbrace{\left(\frac{1}{n}\right)^0}_{=1} + \underbrace{\binom{n}{1}}_{n}\cdot 1^{n-1}\left(\frac{1}{n}\right)^1 + \sum_{k=2}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k$$

$$= 1 + 1 + \sum_{k=2}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k \leq 2 + 1 - \frac{1}{n} \leq 3$$

We have: $\binom{n}{k}\cdot\left(\frac{1}{n}\right)^k = \frac{n!}{(n-k)!\cdot k!}\cdot\left(\frac{1}{n^k}\right) = \frac{n\cdot(n-1)(n-2)\cdots(n-k+1)}{n\cdot n\cdot n\cdots n}\cdot\frac{1}{k!} \leq \frac{1}{k!}$ (with $\frac{n-1}{n} \leq 1$)

$$\leq \frac{1}{k\cdot(k-1)} = \frac{1}{k-1} - \frac{1}{k} \quad \text{and} \quad \sum_{k=2}^{n}\left(\frac{1}{k-1} - \frac{1}{k}\right) \overset{\text{telescoping}}{=} 1 - \frac{1}{n}$$

$\overset{\text{fact}}{\Longrightarrow}$ (Monotone convergence criterion) The sequence $(a_n)_{n\in\mathbb{N}}$ is convergent.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n =: e \qquad \text{Euler's number}$$