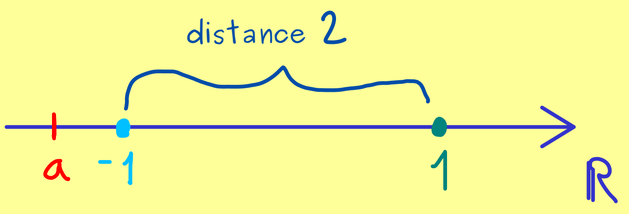
The Bright Side of Mathematics

Real Analysis – Part 3

Example: $(a_n)_{n\in\mathbb{N}} = \left((-1)^n\right)_{n\in\mathbb{N}}$ is divergent.

Proof: Assume the sequence $(a_n)_{n\in\mathbb{N}}$ is convergent to $a \in \mathbb{R}$.

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\; \forall n \geq N : \quad |a_n - a| < \varepsilon$$



Choose: $\varepsilon = 1$ Then: $|a_N - a| < \varepsilon$

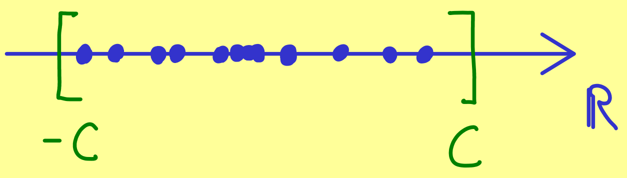
and $|a_{N+1} - a| < \varepsilon$

Hence: $|1 - a| < \varepsilon$ and $|(-1) - a| < \varepsilon$

$$\underline{2} = |1 - (-1)| = |1 - a + a - (-1)| \leq |1 - a| + |a - (-1)| = |1 - a| + |(-1) - a| \underline{< 2} \quad ↯$$

Definition: A sequence $(a_n)_{n\in\mathbb{N}}$ is called bounded if

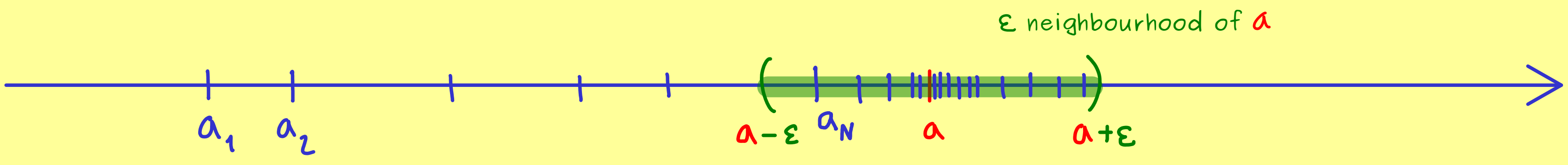
$$\exists C \in \mathbb{R} \;\; \forall n \in \mathbb{N} : \quad |a_n| \leq C$$



Otherwise, the sequence is called unbounded.

Important fact: $(a_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow$ $(a_n)_{n\in\mathbb{N}}$ bounded

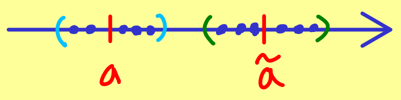
Proof: There is $a \in \mathbb{R}$ with:



$$C := \max\left(|a_1|, |a_2|, |a_3|, \dots, |a_{N-1}|, |a| + \varepsilon\right)$$

Important fact: $(a_n)_{n\in\mathbb{N}}$ convergent $\Rightarrow$ There is only one limit $a \in \mathbb{R}$ $\;\; \left(= \lim\limits_{n\to\infty} a_n\right)$

Proof: Assume there are two limits $a \neq \tilde{a}$. $\varepsilon := \frac{1}{4}|a - \tilde{a}| > 0$



Then: $\exists N \in \mathbb{N} \;\; \forall n \geq N : \quad |a_n - a| < \varepsilon$

$\exists \tilde{N} \in \mathbb{N} \;\; \forall n \geq \tilde{N} : \quad |a_n - \tilde{a}| < \varepsilon$

Therefore: For $n \geq \max(N, \tilde{N})$: $|a - \tilde{a}| = |a - a_n + a_n - \tilde{a}| \leq \overbrace{|a - a_n|}^{<\varepsilon} + \overbrace{|a_n - \tilde{a}|}^{<\varepsilon}$

$$< \frac{1}{2}|a - \tilde{a}| \quad ↯$$