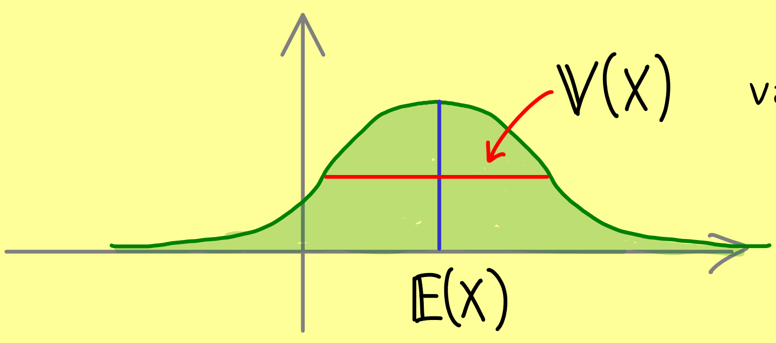
BECOME A MEMBER

ON STEADY

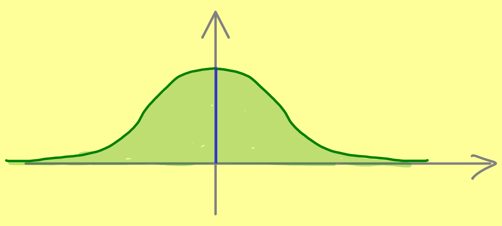
The Bright Side of Mathematics

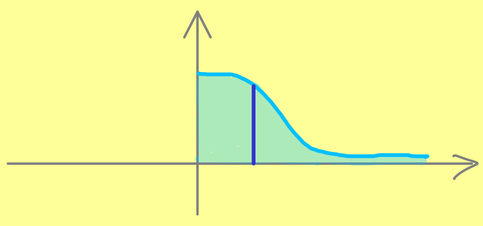
Probability Theory – Part 16



Definition: $(\Omega, \mathcal{A}, \mathbb{P})$ probability space, $X: \Omega \longrightarrow \mathbb{R}$ random variable.

$$\mathrm{Var}(X) := \mathbb{E}\Big(\underbrace{\big(X - \mathbb{E}(X)\big)^2}_{\text{new random variable}}\Big)$$





$$= \mathbb{E}\Big(X^2 - 2\cdot\mathbb{E}(X)\cdot X + \mathbb{E}(X)^2\Big)$$

$$\overset{\text{linearity}}{=} \mathbb{E}(X^2) - 2\cdot\mathbb{E}(X)\,\mathbb{E}(X) + \underbrace{\mathbb{E}\big(\mathbb{E}(X)^2\big)}_{\mathbb{E}(X)^2 \cdot \underbrace{\int_\Omega 1 \, d\mathbb{P}}_{\mathbb{P}(\Omega) = 1}}$$

$$= \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

We need to assume that $\mathbb{E}(X^2) = \int_\Omega X^2 \, d\mathbb{P}$ exists

change-of-variables

$$= \begin{cases} \int_{X(\Omega)} x^2 \cdot f_X(x)\, dx & \text{continuous case} \\ \sum_{x \in X(\Omega)} x^2 \cdot p_x & \text{discrete case} \end{cases}$$

Examples:

(a) $X \sim \text{Uniform}\big(\{x_1, x_2, \dots, x_n\}\big)$ discrete case with $\mathbb{P}_X(\{x_i\}) = \frac{1}{n}$

$$\mathbb{E}(X) = \int_\Omega X \, d\mathbb{P} = \sum_{j=1}^{n} x_j \, \mathbb{P}_X(\{x_j\}) = \underbrace{\frac{1}{n} \sum_{j=1}^{n} x_j}_{= \bar{x}}$$

arithmetic mean

$$\mathrm{Var}(X) = \int_\Omega \big(X - \underbrace{\mathbb{E}(X)}_{\bar{x}}\big)^2 \, d\mathbb{P} = \sum_{j=1}^{n} (x_j - \bar{x})^2 \cdot \mathbb{P}_X(\{x_j\})$$

$$= \frac{1}{n} \cdot \sum_{j=1}^{n} (x_j - \bar{x})^2$$

(b) $X \sim \mathrm{Exp}(\lambda)$ (exponential distribution) $\mathbb{E}(X) = \frac{1}{\lambda}$

$$\mathbb{E}(X^2) = \int_\Omega X^2 \, d\mathbb{P} = \int_{\mathbb{R}} x^2 \cdot f_X(x)\, dx$$

$$\boxed{f_X(x) = \begin{cases} \lambda e^{-\lambda \cdot x} & , x > 0 \\ 0 & , x \leq 0 \end{cases}}$$

$$= \int_0^\infty x^2 \cdot \lambda e^{-\lambda \cdot x} \, dx \overset{\text{integration by parts}}{=} \frac{2}{\lambda^2}$$

$$\mathrm{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{1}{\lambda^2}$$