

The Bright Side of Mathematics



Jordan normal form part 4

$$A \in \mathbb{C}^{n \times n} \Rightarrow A = X J X^{-1}$$

transformation matrix \nearrow X \nearrow J \nearrow X^{-1} Jordan normal form

- Recipe:
- (1) Eigenvalues
 - (2) Alg. and geom. multiplicities
 - (3) Calculate the eigenspace and generalised eigenspaces
 - (4) transformation matrix X

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

\leadsto Find J and X !

(1) Eigenvalues: $\det(A - \lambda I) = (1 - \lambda)^5 \Rightarrow \lambda_1 = 1$

(2) Multiplicities: For $\lambda_1 = 1$: algebraic multiplicity $\alpha(\lambda_1) = 5 \leadsto$ Jordan block of size 5×5

$$\text{Ker}(A - \lambda_1 I) = \text{Ker} \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \xrightarrow{\text{Gaussian elimin.}} \text{Ker} \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

geometric multiplicity $\gamma(\lambda_1) = 2 \leadsto$ two Jordan boxes

(3) Eigenspace and generalised eigenspaces: For $\lambda_1 = 1$:

$$\text{Ker}(A - \lambda_1 I) = \text{Span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right) \leadsto \text{two linearly independent eigenvectors}$$

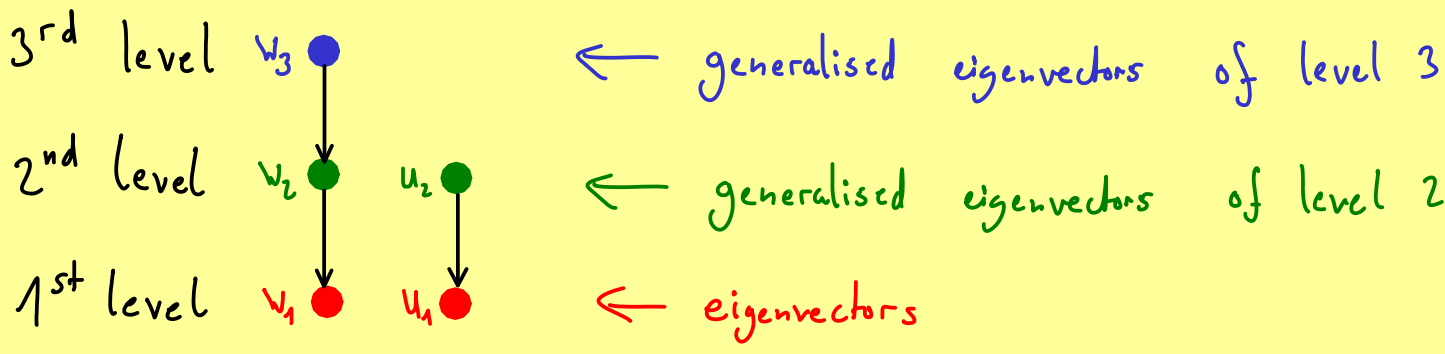
$$\text{Ker}(A - \lambda_1 I)^2 = \text{Ker} \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{Ker} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4-dimensional \downarrow Gaussian elimin.

$$= \text{Ker} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{Span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right)$$

$$\text{Ker}(A - \lambda_1 I)^3 = \text{Ker} \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{Ker} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \text{Ker} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \text{Span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} \right)$$



Choose w_3 : generalised eigenvectors of level 3: $w_3 \in \text{Ker}(A - \lambda_1 I)^3$ but $w_3 \notin \text{Ker}(A - \lambda_1 I)^2$

Then set:

$$w_{k-1} := (A - \lambda_1 I) w_k$$

$$w_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$w_1 = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

Choose u_2 : generalised eigenvectors of level 2:

$$u_2 \in \text{Ker}(A - \lambda_1 I)^2 \text{ but } u_2 \notin \text{Span}(\text{Ker}(A - \lambda_1 I) \cup \{w_2\})$$

$\text{Span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)$

$$\text{Ker}(A - \lambda_1 I) = \text{Span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \right), \quad \text{Ker}(A - \lambda_1 I)^2 = \text{Span} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right), \quad w_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Then set: $u_1 = (A - \lambda_1 I) u_2 = \begin{pmatrix} 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$

(4) transformation matrix:

$$(XJ = AX)$$

$$X = \begin{pmatrix} | & | & | & | & | \\ w_1 & w_2 & w_3 & u_1 & u_2 \\ | & | & | & | & | \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$\underbrace{\hspace{10em}}_{\lambda_1 = 1 \text{ Jordan block}}$

$$\leadsto J = X^{-1} A X = \begin{pmatrix} \boxed{1} & 1 & & & \\ & \boxed{1} & 1 & & \\ & & \boxed{1} & 1 & \\ & & & \boxed{1} & 1 \\ & & & & \boxed{1} \end{pmatrix}$$