Jordan normal form – part 2

$$A = \begin{pmatrix} 3 & 1 & 0 & 1 \\ -1 & 5 & 4 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Find a Jordan normal form for the matrix A.

Recipe:
(1) Eigenvalues of A
(2) Algebraic and geometric multiplicities
(3) Dimensions of generalised eigenspaces

(1) Eigenvalues:

$$\det(A - \lambda \mathbb{1}) = \det \begin{pmatrix} 3-\lambda & 1 & 0 & 1 \\ -1 & 5-\lambda & 4 & 1 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 4-\lambda \end{pmatrix}$$

$$= \det \begin{pmatrix} 3-\lambda & 1 \\ -1 & 5-\lambda \end{pmatrix} \cdot \det \begin{pmatrix} 2-\lambda & 0 \\ 0 & 4-\lambda \end{pmatrix}$$

$$= \big((3-\lambda)(5-\lambda) + 1 \big) \cdot (2-\lambda)(4-\lambda)$$

$$= \big(16 - 8\lambda + \lambda^2 \big) \cdot (2-\lambda) \cdot (4-\lambda)$$

$$= (4-\lambda)^2 \cdot (2-\lambda) \cdot (4-\lambda) = (2-\lambda)^1 \cdot (4-\lambda)^3$$

Eigenvalues $\lambda_1 = 2$ with $\alpha(\lambda_1) = 1 \rightsquigarrow$ Jordan block 1×1

$\lambda_2 = 4$ with $\alpha(\lambda_2) = 3 \rightsquigarrow$ Jordan block 3×3

$$J = \begin{pmatrix} \boxed{2} & & & \\ & 4 & 1 & \\ & & 4 & \\ & & & 4 \end{pmatrix}$$

1^{st} possibility: $\begin{pmatrix} 4 & & \\ & 4 & \\ & & 4 \end{pmatrix}$ $\quad \gamma(\lambda_2) = 3$ (geometric multiplicity)

2^{nd} possibility: $\begin{pmatrix} 4 & 1 & \\ & 4 & \\ & & 4 \end{pmatrix}$ $\quad \gamma(\lambda_2) = 2$

3^{rd} possibility: $\begin{pmatrix} 4 & 1 & \\ & 4 & 1 \\ & & 4 \end{pmatrix}$ $\quad \gamma(\lambda_2) = 1$

Eigenspace for $\lambda_2 = 4$:

$$\mathrm{Ker}(A - \lambda_2 \mathbb{1}) = \mathrm{Ker} \begin{pmatrix} 3-4 & 1 & 0 & 1 \\ -1 & 5-4 & 4 & 1 \\ 0 & 0 & 2-4 & 0 \\ 0 & 0 & 0 & 4-4 \end{pmatrix} = \mathrm{Ker} \begin{pmatrix} \boxed{-1} & 1 & 0 & 1 \\ -1 & 1 & 4 & 1 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\mathrm{II}-\mathrm{I}}{=} \mathrm{Ker} \begin{pmatrix} \boxed{-1} & 1 & 0 & 1 \\ 0 & 0 & \boxed{4} & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\mathrm{III}+\frac{1}{2}\mathrm{II}}{=} \mathrm{Ker} \begin{pmatrix} \boxed{-1} & 1 & 0 & 1 \\ 0 & 0 & \boxed{4} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

x_2, x_4: free variables

$$\gamma(\lambda_2) = \dim\big(\mathrm{Ker}(A - \lambda_2 \mathbb{1})\big) = 2 \Leftarrow$$