BECOME A MEMBER

ON STEADY

The Bright Side of Mathematics

Jordan normal form – part 1

A square matrix diagonalisable $:\Longleftrightarrow$

There is an invertible X with $X^{-1} A X = \mathcal{D}$ diagonal matrix

$A = X \underbrace{\mathcal{D}}_{\text{Jordan normal form}} X^{-1}$ (matrix decomposition)

$A \in \mathbb{C}^{n \times n} \Rightarrow$ There exists a Jordan normal form J for A: $A = X J X^{-1}$

Example: $A \in \mathbb{C}^{9 \times 9}$ and with eigenvalues: $\{ \underbrace{2,2,2}_{\substack{\text{algebraic}\\\text{multiplicity}\\3}}, \underbrace{3,3,3,3}_{\substack{\text{algebraic}\\\text{multiplicity}\\4}}, \underbrace{4,4}_{\substack{\text{algebraic}\\\text{multiplicity}\\2}} \}$

$$J = \begin{pmatrix} \boxed{\begin{matrix} 2 & & (*) \\ & 2 & \\ & & 2 \end{matrix}} & & \\ & \boxed{\begin{matrix} 3 & & & \\ & 3 & (*) & \\ & & 3 & \\ & & & 3 \end{matrix}} & \\ & & \boxed{\begin{matrix} 4 & (*) \\ & 4 \end{matrix}} \end{pmatrix}$$

three Jordan blocks

$\boxed{\begin{matrix} 2 & & (*) \\ & 2 & \\ & & 2 \end{matrix}}$:

(1) $\begin{pmatrix} 2 & & \\ & 2 & \\ & & 2 \end{pmatrix}$ three Jordan boxes, geometric multiplicity: 3

(2) $\begin{pmatrix} 2 & 1 & \\ & 2 & \\ & & 2 \end{pmatrix}$ two Jordan boxes, geometric multiplicity: 2

(3) $\begin{pmatrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{pmatrix}$ one Jordan boxes, geometric multiplicity: 1

$\boxed{\begin{matrix} 3 & & & \\ & 3 & (*) & \\ & & 3 & \\ & & & 3 \end{matrix}}$:

(1) $\begin{pmatrix} 3 & & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$ four Jordan boxes, geometric multiplicity 4

(2) $\begin{pmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & \\ & & & 3 \end{pmatrix}$ three Jordan boxes, geometric multiplicity 3

(3) $\begin{pmatrix} 3 & 1 & & \\ & 3 & & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$ (4) $\begin{pmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & \\ & & & 3 \end{pmatrix}$ two Jordan boxes, geometric multiplicity 2

(5) $\begin{pmatrix} 3 & 1 & & \\ & 3 & 1 & \\ & & 3 & 1 \\ & & & 3 \end{pmatrix}$ one Jordan boxes, geometric multiplicity 1

$\boxed{X J X^{-1} = A}$

Recipe: (1) Calculate eigenvalues of A: $\lambda_1, \lambda_2, \dots, \lambda_k$

For $i = 1..k$: (2) Determine algebraic multiplicity of λ_i and geometric multiplicity of $\lambda_i := \dim(\operatorname{Ker}(A - \lambda_i \mathbb{1}))$

(3) If needed: $\dim(\operatorname{Ker}(A - \lambda_i \mathbb{1})^2)$, $\dim(\operatorname{Ker}(A - \lambda_i)^3) \dots$