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ON STEADY

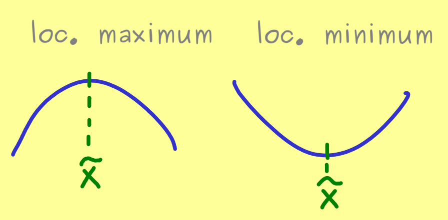
The Bright Side of
Mathematics

Multivariable Calculus – Part 12

Finding local extrema: one-dimensional case: $f: \mathbb{R} \to \mathbb{R}$

Necessary: If $f: \mathbb{R} \to \mathbb{R}$ is differentiable $\tilde{x} \in \mathbb{R}$ and f has local extremum at $\tilde{x} \in \mathbb{R}$,

then: $f'(\tilde{x}) = 0$



Sufficient: If $f: \mathbb{R} \to \mathbb{R}$ is two-times differentiable and $f'(\tilde{x}) = 0$,

then: (a) $f''(\tilde{x}) < 0 \implies f$ has local maximum at $\tilde{x}$

(b) $f''(\tilde{x}) > 0 \implies f$ has local minimum at $\tilde{x}$

Definition: Let $f: \mathbb{R}^n \to \mathbb{R}$ be partially differentiable, so $\frac{\partial f}{\partial x_i}: \mathbb{R}^n \to \mathbb{R}$ defines a new function for each $i \in \{1, \dots, n\}$.

For $\tilde{x} \in \mathbb{R}^n$:

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}\right)(\tilde{x}) = \lim_{h \to 0} \frac{\frac{\partial f}{\partial x_2}(\tilde{x}_1 + h, \tilde{x}_2, \dots, \tilde{x}_n) - \frac{\partial f}{\partial x_2}(\tilde{x}_1, \tilde{x}_2, \dots, \tilde{x}_n)}{h}$$

$$=: \frac{\partial^2 f}{\partial x_1 \partial x_2}(\tilde{x})$$

$$\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_1}\right)(\tilde{x}) =: \frac{\partial^2 f}{\partial x_1^2}(\tilde{x})$$

second-order partial derivatives of f at $\tilde{x}$

Example: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = \sin(x_1 \cdot x_2)$

$$\frac{\partial f}{\partial x_1}(x) = x_2 \cdot \cos(x_1 \cdot x_2), \qquad \frac{\partial f}{\partial x_2}(x) = x_1 \cdot \cos(x_1 \cdot x_2)$$

$$\frac{\partial^2 f}{\partial x_1^2}(x) = -x_2^2 \sin(x_1 \cdot x_2), \qquad \frac{\partial^2 f}{\partial x_2^2}(x) = -x_1^2 \sin(x_1 \cdot x_2)$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1}(x) = \cos(x_1 \cdot x_2) - x_1 \cdot x_2 \cdot \sin(x_1 \cdot x_2)$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2}(x) = \cos(x_1 \cdot x_2) - x_2 \cdot x_1 \cdot \sin(x_1 \cdot x_2)$$