The Bright Side of Mathematics

Multivariable Calculus – Part 8

Gradient: $f: \mathbb{R}^n \to \mathbb{R}$ (totally) differentiable at $\tilde{x} \in \mathbb{R}^n$

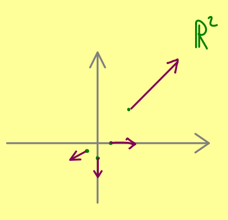
Jacobian matrix: $J_f(\tilde{x}) = \left(\frac{\partial f}{\partial x_1}(\tilde{x}) \quad \frac{\partial f}{\partial x_2}(\tilde{x}) \quad \cdots \quad \frac{\partial f}{\partial x_n}(\tilde{x}) \right) \in \mathbb{R}^{1 \times n}$

Gradient: $$\nabla f(\tilde{x}) = \operatorname{grad} f(\tilde{x}) := \begin{pmatrix} \frac{\partial f}{\partial x_1}(\tilde{x}) \\ \vdots \\ \frac{\partial f}{\partial x_n}(\tilde{x}) \end{pmatrix} \in \mathbb{R}^n$$

(∇: nabla)

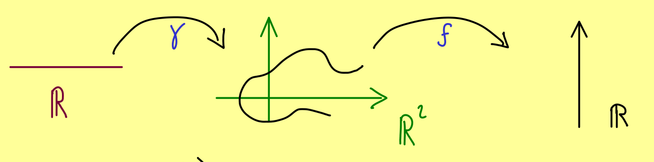
Example: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x_1, x_2) = x_1^2 + x_2^2$

$$\operatorname{grad} f(x_1, x_2) = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$



Application of the chain rule:

$\gamma: \mathbb{R} \to \mathbb{R}^2$ $\qquad$ $f: \mathbb{R}^2 \to \mathbb{R}$



$\gamma(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$ $\qquad$ $f(x_1, x_2) = x_1^2 + x_2^2$

$$J_{f \circ \gamma}(t) = J_f(\gamma(t)) \cdot J_\gamma(t) = \left(2\cos(t), 2\sin(t) \right) \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$$

$$= \left(0 \right)$$

Rewrite it: $$J_f(\gamma(t)) \cdot J_\gamma(t) = \left\langle \operatorname{grad} f(\gamma(t)), \gamma'(t) \right\rangle$$

(standard inner product)

$$= 0$$

orthogonality!