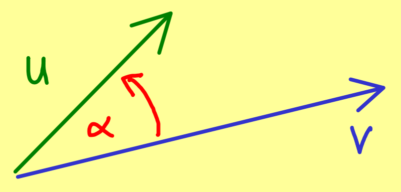
The Bright Side of Mathematics

Linear Algebra – Part 9

inner product and norm in $\mathbb{R}^n$?

↳ give more structure to the vector space

↳ we can do geometry (measure angles and lengths)



Definition: For $u, v \in \mathbb{R}^n$, we define:

$$\langle u, v \rangle := u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i \qquad \text{(standard) inner product}$$

If $\langle u, v \rangle = 0$, we say that u, v are orthogonal.

Properties: The map $\langle \cdot , \cdot \rangle : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ has the following properties:

(1) $\langle u, u \rangle \geq 0$ for all $u \in \mathbb{R}^n$
$\langle u, u \rangle = 0 \iff u = 0$ (positive definite)

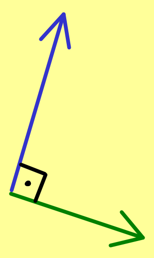
(2) $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in \mathbb{R}^n$ (symmetric)

(3) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$
$\langle u, \lambda \cdot v \rangle = \lambda \cdot \langle u, v \rangle$ (linear in the 2nd argument)
for all $u, v, w \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$

Definition: For $u \in \mathbb{R}^n$, we define:

$$\|u\| := \sqrt{\langle u, u \rangle} = \sqrt{u_1^2 + u_2^2 + \cdots + u_n^2} \qquad \underset{\text{(standard) norm}}{\overset{\text{Euclidean}}{/\!/}}$$

Example: $u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^4$, $v = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix} \in \mathbb{R}^4$, $\langle u, v \rangle = 0$



$\|u\| = \sqrt{1^2 + 1^2} = \sqrt{2}$, $\|v\| = \sqrt{2^2} = 2$