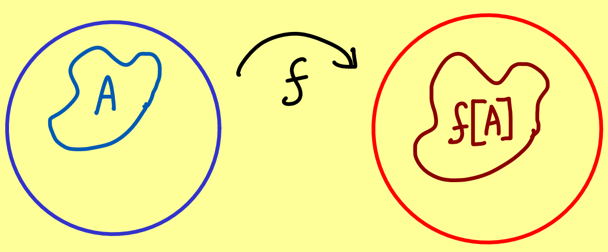
The Bright Side of
Mathematics

Functional analysis – part 26

Open mapping theorem (Banach–Schauder theorem)

What is an open map?



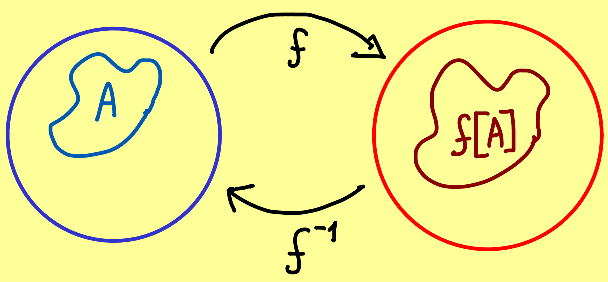
Let (X, d_X), (Y, d_Y) be two metric spaces.

$f: X \to Y$ is called open if

$A \subseteq X$ open in $X \Rightarrow f[A] \subseteq Y$ open in Y

General example: If $f: X \to Y$ is bijective and $f^{-1}: Y \to X$ is continuous, then:

$f: X \to Y$ is an open map



Continuity of f^{-1}: $A \subseteq X$ open in $X \Rightarrow \underbrace{(f^{-1})^{-1}[A]}_{f[A]} \subseteq Y$ open in Y

Examples: (a) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^3$ open

(b) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$ not open $A = (-2, 2) \rightsquigarrow f[A] = [0, 4)$

Open Mapping Theorem: Let X, Y be Banach spaces. For $T \in \mathcal{B}(X, Y)$ holds:

T surjective $\Leftrightarrow$ T open map