The Bright Side of Mathematics

Functional analysis – part 22

Dual spaces: X normed space

X' normed space

$$X' := \{\, \ell : X \to \mathbb{F} \mid \ell \text{ linear + bounded} \,\}$$

Recall the Riesz representation theorem: X Hilbert space. Then: $X' \overset{\text{isometric}}{\underset{\text{isomorphism}}{\rightleftarrows}} X$

Proposition: Let X be a normed space. Then $(X', \|\cdot\|_{X\to\mathbb{F}})$ is a Banach space.

Proof: Let $(\ell_k)_{k\in\mathbb{N}} \subseteq X'$ be a Cauchy sequence:

$$\forall \varepsilon > 0 \quad \exists N \in \mathbb{N} \quad \forall n,m \geq N : \quad \|\ell_n - \ell_m\|_{X\to\mathbb{F}} < \varepsilon$$

$$\|\ell_n - \ell_m\|_{X\to\mathbb{F}} \geq \frac{1}{\|x\|_X} |\ell_n(x) - \ell_m(x)| \quad \text{for } x \in X,\ x \neq 0.$$

$\Rightarrow (\ell_k(x))_{k\in\mathbb{N}} \subseteq \mathbb{F}$ is Cauchy sequence for all $x \in X$.

$$\Rightarrow \quad \ell(x) := \lim_{k\to\infty} \ell_k(x) \quad , \quad \ell : X \to \mathbb{F}$$

Show: (1) ℓ is linear ✓

(2) ℓ is bounded ✓

(3) $\|\ell_k - \ell\|_{X\to\mathbb{F}} \xrightarrow{k\to\infty} 0$ ✓

For (2): $\|\ell_n\|_{X\to\mathbb{F}} \leq \underbrace{\|\ell_n - \ell_N\|_{X\to\mathbb{F}}}_{<\varepsilon} + \underbrace{\|\ell_N\|_{X\to\mathbb{F}}}_{=:C} \leq C + \varepsilon$ for all $n \geq N$

$$\Rightarrow \quad |\ell(x)| = \left|\lim_{k\to\infty} \ell_k(x)\right| = \lim_{k\to\infty} |\ell_k(x)| \leq \lim_{k\to\infty} \underbrace{\|\ell_k\|_{X\to\mathbb{F}}}_{\leq \tilde{C}} \|x\|_X$$

$$\Rightarrow \quad \|\ell\|_{X\to\mathbb{F}} \leq \tilde{C} < \infty$$

For (3): For $\varepsilon > 0$ choose $N \in \mathbb{N}$ such that for all $n,m \geq N$:

$$\frac{1}{\|x\|_X} |\ell_n(x) - \ell_m(x)| < \varepsilon$$

$$\Rightarrow \quad \sup_{\substack{x\in X \\ x\neq 0}} \frac{1}{\|x\|_X} \Big|\ell_n(x) - \overbrace{\lim_{m\to\infty} \ell_m(x)}^{\ell(x)}\Big| \leq \varepsilon \quad \Rightarrow \quad \|\ell_n - \ell\|_{X\to\mathbb{F}} \leq \varepsilon$$