BECOME A MEMBER

ON STEADY

The Bright Side of
Mathematics

Functional analysis – part 18

Compact operators:

$T: \mathbb{F}^n \longrightarrow \mathbb{F}^m$ linear (standard norm)

$\Rightarrow$ T is continuous / bounded

$\Rightarrow$ $T[\mathcal{B}_1(0)] \subseteq \mathbb{F}^m$ bounded

$\Rightarrow$ $\overline{T[\mathcal{B}_1(0)]} \subseteq \mathbb{F}^m$ compact

However: $I: \ell^p(\mathbb{N}) \longrightarrow \ell^p(\mathbb{N})$, $p \in [1,\infty)$,

$x \longmapsto x$ $\Rightarrow$ $\overline{I[\mathcal{B}_1(0)]} = \overline{\mathcal{B}_1(0)}$ not compact

closed unit ball in $\ell^p(\mathbb{N})$

Definition: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be two normed spaces. A bounded linear operator $T: X \to Y$ is called compact if $\overline{T[\mathcal{B}_1(0)]} \subseteq Y$ is a compact set.

Example: Integral operator $T_k: C([0,1]) \longrightarrow C([0,1])$ for $k \in C([0,1]\times[0,1])$

with supremum norm $\|\cdot\|_\infty$

$$(T_k f)(s) := \int_0^1 k(s,t)\, f(t)\, dt$$

Fact: k is uniformly continuous:

$$\forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{(s_1,t_1),(s_2,t_2)} \quad \|(s_1,t_1)-(s_2,t_2)\| < \delta \Rightarrow |k(s_1,t_1)-k(s_2,t_2)| < \varepsilon$$

For $\varepsilon > 0$, choose $\delta > 0$ such that . Therefore for $s_1, s_2 \in [0,1]$ with $|s_1 - s_2| < \delta$:

$$|(T_k f)(s_1) - (T_k f)(s_2)| = \left|\int_0^1 \Big(k(s_1,t)\, f(t) - k(s_2,t)\, f(t)\Big) dt\right|$$

$$\leq \int_0^1 \underbrace{|k(s_1,t) - k(s_2,t)|}_{<\varepsilon} \cdot \underbrace{|f(t)|}_{\leq \|f\|_\infty} \, dt < \varepsilon \cdot \|f\|_\infty$$

$A := T_k[\mathcal{B}_1(0)]$. We have:

$$\forall_{\varepsilon>0} \ \exists_{\delta>0} \ \forall_{s_1,s_2\in[0,1]} \ \forall_{g\in A} : |s_1 - s_2| < \delta \Rightarrow |g(s_1) - g(s_2)| < \varepsilon$$

$\Rightarrow$ $T_k[\mathcal{B}_1(0)]$ is uniformly equicontinuous

Boundedness:

$$\|T_k\| = \sup\left\{ \|T_k f\|_\infty \;\middle|\; \|f\|_\infty = 1 \right\}$$

$$= \sup\left\{ \sup_{s\in[0,1]} \left|\int_0^1 k(s,t) f(t)\, dt\right| \;\middle|\; \|f\|_\infty = 1 \right\}$$

$$\leq \sup\left\{ \sup_{s\in[0,1]} \int_0^1 |k(s,t)| \underbrace{|f(t)|}_{\leq\|f\|_\infty} dt \;\middle|\; \|f\|_\infty = 1 \right\}$$

$$\leq \sup_{s\in[0,1]} \int_0^1 |k(s,t)|\, dt \leq \|k\|_\infty$$

$\Rightarrow$ By Arzelà-Ascoli: $\overline{T_k[\mathcal{B}_1(0)]}$ is compact $\Rightarrow$ T_k compact operator