



## The Bright Side of Mathematics

### Functional analysis - part 11

Orthogonality: Let  $(X, \langle \cdot, \cdot \rangle)$  be an inner product space.

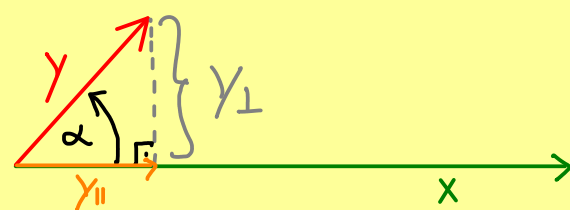
(a)  $x, y \in X$  are called orthogonal if  $\langle x, y \rangle = 0$ . Write  $x \perp y$ .

(b) For  $U, V \subseteq X$ , write  $U \perp V$  if  $x \perp y$  for all  $x \in U, y \in V$ .

(c) For  $U \subseteq X$ , the orthogonal complement of  $U$  is

$$U^\perp := \{x \in X \mid \langle x, u \rangle = 0 \text{ for all } u \in U\}$$

*is always a subspace in  $X$*



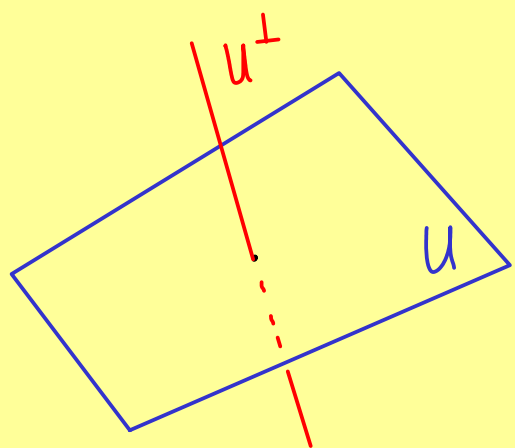
Remark: (1)  $\{0\}^\perp = X$ ,  $X^\perp = \{0\}$

(2)  $U \subseteq V \Rightarrow U^\perp \supseteq V^\perp$

Proof:  $x \in V^\perp \Rightarrow \langle x, v \rangle = 0$  for all  $v \in V$

$\stackrel{U \subseteq V}{\Rightarrow} \langle x, u \rangle = 0$  for all  $u \in U \Rightarrow x \in U^\perp$

(3) If  $x \perp y$ , then  $\|x+y\|_{\langle \cdot, \cdot \rangle}^2 = \|x\|_{\langle \cdot, \cdot \rangle}^2 + \|y\|_{\langle \cdot, \cdot \rangle}^2$  (Pythagorean theorem)



$U^\perp$  is always closed