Problem 2 *Estimate the radius of convergence* (4 points)
The sequence $(a_k)_{k\in\mathbb{N}}$ is given by

$$a_k := \begin{cases} 2^{-\frac{k+1}{2}}, & k \text{ odd}, \\ 3^{-\frac{k}{2}}, & k \text{ even}. \end{cases}$$

We consider the power series $\sum_{k=0}^{\infty} a_k x^k$ and want to calculate its radius of convergence.

a) Can you find the radius of convergence by using the ratio test (quotient formula from the tutorial)? Does the ratio test give you, at least, some numbers $a, b \in [0, \infty]$ with $a \leq R \leq b$?

b) Calculate the radius of convergence by using the *Cauchy-Hadamard theorem* (root formula from the tutorial).

Solutions

$$a_k = \begin{cases} 2^{-\frac{k+1}{2}} & , k \text{ odd} \\ 3^{-\frac{k}{2}} & , k \text{ even} \end{cases}$$

(a) Ratio test:

$$\frac{|a_k|}{|a_{k+1}|} = \begin{cases} \dfrac{2^{-\frac{k+1}{2}}}{3^{-\frac{k+1}{2}}} & , k \text{ odd} \\ \dfrac{3^{-\frac{k}{2}}}{2^{-\frac{(k+2)}{2}}} & , k \text{ even} \end{cases}$$

$$= \begin{cases} \left(\frac{2}{3}\right)^{-\frac{k+1}{2}} & , k \text{ odd} \\ \left(\frac{3}{2}\right)^{-\frac{k}{2}} \cdot \frac{1}{2} & , k \text{ even} \end{cases} \quad \rightarrow \text{ not convergent}$$

$$\limsup_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = \infty \quad \text{and} \quad \liminf_{k\to\infty} \left|\frac{a_k}{a_{k+1}}\right| = 0$$

Ratio test gives us no information about the conv. radius!

(b) Cauchy-Hadamard:

$$R^{-1} = \limsup_{k\to\infty} \sqrt[k]{|a_k|}$$

$$\sqrt[k]{|a_k|} = \begin{cases} \left(2^{-\frac{k+1}{2}}\right)^{\frac{1}{k}} & , \ k \text{ odd} \\ \left(3^{-\frac{k}{2}}\right)^{\frac{1}{k}} & , \ k \text{ even} \end{cases}$$

$$= \begin{cases} 2^{-\frac{1}{2}-\frac{1}{2k}} & , \ k \text{ odd} \\ 3^{-\frac{1}{2}} & , \ k \text{ even} \end{cases}$$

$\left(\sqrt{3} > \sqrt{2} \iff 2^{-\frac{1}{2}} > 3^{-\frac{1}{2}}\right)$

$$R^{-1} = \limsup_{k\to\infty} \sqrt[k]{|a_k|} = 2^{-\frac{1}{2}} \implies \underline{R = \sqrt{2}}$$