Problem 2 *Continuous functions, maximum and minimum* (4 points)

a) Consider two functions $f, g : (0,1) \to \mathbb{R}$ defined by $f(x) = x(1-x)$ and $g(x) = \frac{1}{f(x)}$. Show that f and g are continuous by using the continuity definition with sequences (see tutorial).

b) Show that $f : (0,1) \to \mathbb{R}$ defined by $f(x) = x(1-x)$ is continuous by explicitly using the ε-δ-criterion (see tutorial or Theorem 4.16).

c) Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function with $\lim_{x\to\pm\infty} h(x) = 0$. Show that the function has a maximum or a minimum.

Hint: You may find Theorem 4.19 very helpful.

Remark: We say that a function $f : I \to \mathbb{R}$ has a maximum *if there is an $a \in I$ with $f(a) = \sup_{x\in I} f(x)$ and we say that f has a* minimum *if there is a $b \in I$ with $f(b) = \inf_{x\in I} f(x)$.*

d) Is it possible that there is a function $k : [0,1) \to \mathbb{R}$ that is continuous and bounded but that has neither a maximum nor a minimum? Give a short reasoning or just a sketch of a suitable graph.

Solutions

$f, g : (0,1) \to \mathbb{R}, \quad f(x) = x\cdot(1-x), \quad g(x) = \frac{1}{x(1-x)}$

(a) Let $(x_n)_{n\in\mathbb{N}} \subseteq (0,1)$ a sequence with limit $a \in (0,1)$.

Then $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_n(1-x_n) \underset{\text{limit theorems}}{=} \lim_{n\to\infty} x_n \left(1 - \lim_{n\to\infty} x_n\right)$

$= a\cdot(1-a) = f(a) \implies$ continuous at a.

And $\lim_{n\to\infty} g(x_n) \underset{\substack{\text{limit theorems} \\ \text{denominator} \neq 0 \\ \text{and } a \neq 0}}{=} \frac{1}{\lim_{n\to\infty} x_n(1-x_n)} = \frac{1}{f(a)} = g(a)$

$\implies$ continuous at a.

$\implies f, g$ continuous functions.

(b) For f: Let $a \in (0,1)$.

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{3}$

Then for all $x \in (0,1)$ with $|x-a| < \delta$ we have:

$$|f(x) - f(a)| = |x(1-x) - a(1-a)| \qquad (*)$$

$$= |x - x^2 - a + a^2|$$

$$= |x - a + a^2 - x^2| \overset{\Delta\text{-inequ.}}{\leq} |x-a| + |x^2 - a^2|$$

$$= |x-a| + |(x-a)(x+a)|$$

$$= |x-a| + |x-a| \cdot \underbrace{|x+a|}_{\leq 2\text{, since } x,a \in (0,1)}$$

$$\leq 3 \cdot |x-a| < 3 \cdot \delta = \varepsilon$$

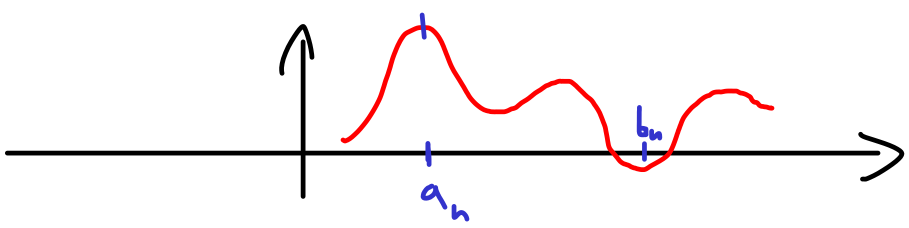
$\Rightarrow f$ is continuous at a. $\Rightarrow f$ is continuous.

(c) Claim: h cont. + vanishes at $\pm\infty$ $\Rightarrow$ max h or min h exists

Proof: Define for each $n \in \mathbb{N}$ the functions:

$$h_n : [-n, n] \to \mathbb{R}, \quad h_n(x) := h(x)$$

By Theorem 4.19, there is an $a_n, b_n \in [-n, n]$ with $\max h_n = h_n(a_n)$, $\min h_n = h_n(b_n)$

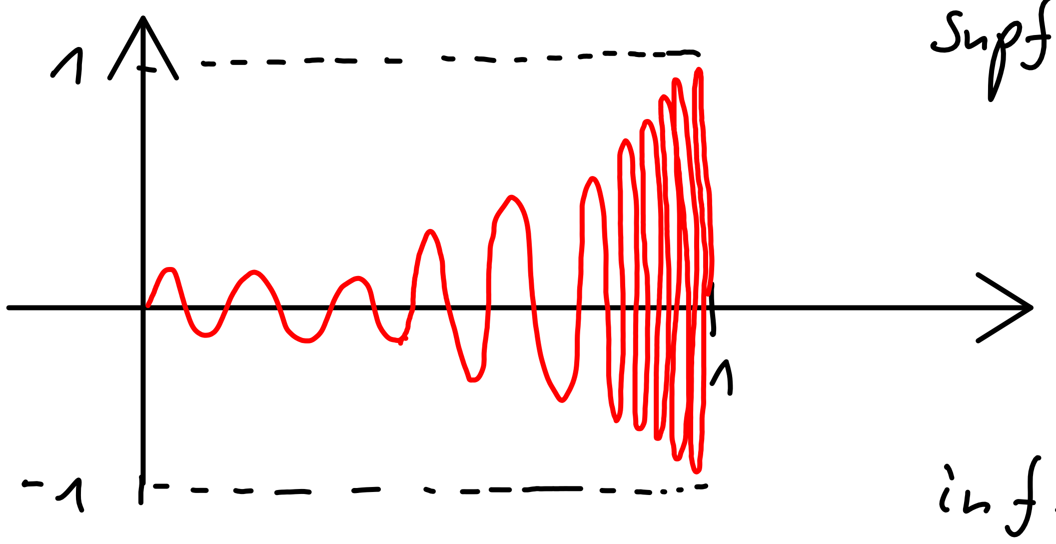


$(h_n(a_n) = h(a_n))$

$\Rightarrow$ h has a minimum.

Third possibility: $\lim_{n\to\infty} h(a_n) > 0$, analogously!

(d) Such a function $k: (0,1) \to \mathbb{R}$ can exist:



$\sup f = 1$

$\inf f = -1$

$\Rightarrow$ no max! no min!