Problem 3 *Convergent and absolutely convergent series* (4 points)
Investigate whether the following series are convergent and even absolutely convergent:

a) $\sum_{n=1}^{\infty} \binom{3n}{2n} 2^{-7n+5}$

b) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{3n+1}{5n^2} + \frac{i}{\sqrt{n}} \right)$

c) $\sum_{n=1}^{\infty} \frac{4n+2+in^3}{\sqrt{n}+7-in^4}$

d) $\sum_{n=1}^{\infty} \frac{\left(-\sqrt[n]{n^2}\right)^n}{1+n^2}$

Solutions

(a)
$$\frac{(3n+3)!}{(2n+2)!} \cdot \frac{2^5 \cdot (2n)! \; n! \; 2^{7n}}{(n+1)! \, 2^{7n} \cdot 2^7 \; (3n)! \; 2^5} = \frac{(3n+3)(3n+2)(3n+1)}{(2n+2)(2n+1) \cdot 2^7 \cdot n}$$

$$\xrightarrow{n \to \infty} \frac{3^3}{2^9} < 1$$

$$\Longrightarrow \quad \sum_{n=1}^{\infty} \binom{3n}{2n} 2^{-7n+5}$$ absolutely convergent by ratio test

(b)
$$\sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{3n+1}{5n^2}}_{\xrightarrow{n\to\infty} 0}$$ monotonically

and

$$i \cdot \sum_{n=1}^{\infty} (-1)^n \underbrace{\frac{1}{\sqrt{n}}}_{\xrightarrow{n\to\infty} 0}$$ monotonically

Leibniz!

$\Longrightarrow$ adding both gives us also a convergent series but not abs. convergent

(c) Try minorant criterion $\left| \frac{4n + 2 + in^3}{\sqrt{n} + 7 - in^4} \right| \geq C \cdot \frac{1}{n}$

for a fixed $C \in \mathbb{R}$

$$\Longleftrightarrow \left| \frac{4\frac{1}{n^2} + 2\frac{1}{n^3} + i}{\frac{1}{n^{\frac{7}{2}}} + 7\frac{1}{n^4} + i} \right| \geq C$$

$$\xrightarrow{n \to \infty} 1$$

Left-hand side is convergent to with limit 1. This means that there is a $N \in \mathbb{N}$ such that

$$\left| \frac{4\frac{1}{n^2} + 2\frac{1}{n^3} + i}{\frac{1}{n^{\frac{7}{2}}} + 7\frac{1}{n^4} + i} \right| \geq \underbrace{\frac{1}{2}}_{\overset{!}{=} C} \quad \text{for all} \quad n \geq N$$

$$\Longrightarrow \sum_{n=N}^{\infty} \frac{4n + 2 + in^3}{\sqrt{n} + 7 - in^4}$$ divergent by minorant criterion

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{4n + 2 + in^3}{\sqrt{n} + 7 - in^4}$$ also divergent
(since we only add finitely many terms)

(d) $$\frac{\left(-\sqrt[n]{n^2}\right)^n}{1 + n^2} = (-1)^n \frac{n^2}{1 + n^2}$$ not convergent to zero!

$$\Longrightarrow \sum_{n=1}^{\infty} \frac{\left(-\sqrt[n]{n^2}\right)^n}{1 + n^2}$$ not convergent!