Problem 2 *Limits of series* (4 points)
Consider the partial sums of the following series and show that they converge. Calculate also the value of the series.

a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{4k^2-1}$

b) $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2k^2+4k+2}$

c) $\displaystyle\sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} \left(-\frac{1}{3}\right)^j \right)$

d) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}$

Hint: You are allowed to use $\displaystyle\sum_{k=1}^{\infty} k^{-2} = \frac{\pi^2}{6}$.

Solutions:

(a) First determine the zeros of $4k^2-1$!
$(k = \frac{1}{2}$ and $k = -\frac{1}{2})$

$$\frac{1}{4k^2-1} = \frac{A}{k-\frac{1}{2}} + \frac{B}{k+\frac{1}{2}} \quad \text{(partial fractions)}$$

$$\Rightarrow \frac{k^2-\frac{1}{4}}{4k^2-1} = A\cdot\left(k+\tfrac{1}{2}\right) + B\left(k-\tfrac{1}{2}\right)$$

putting zeros in

$$\Rightarrow \frac{1}{4} = A \quad \text{and} \quad \frac{1}{4} = -B$$

This means: $$\frac{1}{4k^2-1} = \frac{\frac{1}{4}}{k-\frac{1}{2}} - \frac{\frac{1}{4}}{k+\frac{1}{2}}$$

Then the partial sum is given by:

$$\sum_{k=1}^{N} \frac{1}{4k^2-1} = \frac{1}{4}\left(\sum_{k=1}^{N} \frac{1}{k-\frac{1}{2}} - \sum_{k=1}^{N} \frac{1}{k+\frac{1}{2}} \right)$$

$$= \frac{1}{4}\left(\sum_{k=1}^{N} \frac{1}{k-\frac{1}{2}} - \sum_{m=2}^{N+1} \frac{1}{m-\frac{1}{2}} \right) \qquad (m=k+1)$$

$$= \frac{1}{4}\left(\frac{1}{1-\frac{1}{2}} - \frac{1}{(N+1)-\frac{1}{2}} \right) = \frac{1}{2} - \frac{\frac{1}{4}}{N+\frac{1}{2}}$$

$$\xrightarrow{N\to\infty} \frac{1}{2}$$

(b) $$\frac{1}{2k^2+4k+2} = \frac{1}{2}\cdot\frac{1}{(k+1)^2}$$

Hence: $$\sum_{k=0}^{N} \frac{1}{2k^2+4k+2} = \frac{1}{2}\sum_{k=0}^{N} \frac{1}{(k+1)^2}$$

$$= \frac{1}{2}\sum_{m=1}^{N+1} \frac{1}{k^2} \xrightarrow{N\to\infty} \underline{\frac{1}{2}\cdot\frac{\pi^2}{6}}$$

(c) $$\sum_{j=0}^{k} \binom{k}{j}\left(-\frac{1}{3}\right)^j \overset{\text{binomial theorem}}{=} \left(1-\frac{1}{3}\right)^k = \left(\frac{2}{3}\right)^k$$

Hence: $$\sum_{j=0}^{N}\sum_{j=0}^{k} \binom{k}{j}\left(-\frac{1}{3}\right)^j = \sum_{j=0}^{N}\left(\frac{2}{3}\right)^k$$

$$\xrightarrow{N\to\infty} \frac{1}{1-\frac{2}{3}} = \underline{3}$$

(d)

$$\sum_{k=1}^{N} \frac{k}{2^k} = \sum_{k=1}^{N} \left(\frac{2k - k + 2 - 2}{2^k} \right) = \sum_{k=1}^{N} \left(\frac{-k - 2}{2^k} + \frac{2k + 2}{2^k} \right)$$

$m = k-1$

$$= \sum_{k=1}^{N} \frac{k+1}{2^{k-1}} - \sum_{k=1}^{N} \frac{k+2}{2^k} = \sum_{m=0}^{N-1} \frac{m+2}{2^m} - \sum_{k=1}^{N} \frac{k+2}{2^k}$$

$$= \frac{0+2}{2^0} - \frac{N+2}{2^N} = 2 - \frac{N+2}{2^N}$$

$$\xrightarrow{N \to \infty} \underline{2}$$