Problem 3 *Subsequences* (4 points)
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of complex (or real) numbers.

a) Show the following: If both subsequences $(a_{2k})_{k\in\mathbb{N}}$ and $(a_{2k-1})_{k\in\mathbb{N}}$ converge to the same limit, then also $(a_n)_{n\in\mathbb{N}}$ is convergent.

b) Show or give a counterexample: If the three subsequences $(a_{2k})_{k\in\mathbb{N}}$, $(a_{2k-1})_{k\in\mathbb{N}}$ and $(a_{5k})_{k\in\mathbb{N}}$ converge, then also $(a_n)_{n\in\mathbb{N}}$ is convergent.

c) Show or give a counterexample: If the three subsequences $(a_{2k})_{k\in\mathbb{N}}$, $(a_{2k-1})_{k\in\mathbb{N}}$ and $(a_{6k})_{k\in\mathbb{N}}$ converge, then also $(a_n)_{n\in\mathbb{N}}$ is convergent.

d) Show or give a counterexample: If the sequence of the absolute values $(|a_n|)_{n\in\mathbb{N}}$ converges to zero, then also $(a_n)_{n\in\mathbb{N}}$ is convergent.

Solutions

(a) Claim: (a_{2k}) and (a_{2k-1}) have same limit a

$$\Rightarrow (a_n) \text{ is convergent}$$

Proof: Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that

$|a_{2k} - a| < \varepsilon$ and $|a_{2k-1} - a| < \varepsilon$ for all $k \geq N$.

Then:
$$|a_n - a| = \begin{cases} |a_{2k} - a| , & \underset{\text{even}}{n = 2k} \\ |a_{2k-1} - a| , & \underset{\text{odd}}{n = 2k-1} \end{cases}$$

$$< \varepsilon \quad \text{for all} \quad n \geq N \quad (n \geq k)$$

Therefore (a_n) converges to a. $\square$

(b)

Claim: (a_{2k}), (a_{2k-1}) and (a_{5k}) converges $\Rightarrow (a_n)$ is convergent

Proof: Denote the limits:

$$\lim_{k\to\infty} a_{2k} = \tilde{a} \quad , \quad \lim_{k\to\infty} a_{2k-1} = \tilde{\tilde{a}}$$

$$\lim_{k\to\infty} a_{5k} = \tilde{a}$$

The subsequence $(a_{5(2\ell)})_{\ell\in\mathbb{N}}$ is a subsequence of (a_{2k}) and (a_{5k}). Therefore it converges to $\tilde{a}$ and a.

$\Rightarrow \tilde{a} = a$

The subsequence $(a_{5(2\ell-1)})_{\ell\in\mathbb{N}}$ is a subsequence of (a_{2k-1}) and (a_{5k}). Therefore it converges to $\tilde{\tilde{a}}$ and a.

$\Rightarrow \tilde{\tilde{a}} = a \qquad \Rightarrow \tilde{a} = \tilde{\tilde{a}} \overset{(a)}{\Rightarrow} (a_n)$ converges $\square$

(c) Counterexample: $a_n = \begin{cases} 1, & n \text{ even} \\ -1, & n \text{ odd} \end{cases}$

(d) <u>Claim:</u> (a_n) converges to zero $\Longleftrightarrow$ $(|a_n|)_{n\in\mathbb{N}}$ converges to zero.

<u>Proof:</u>

(a_n) converges to zero

$$\Longleftrightarrow \forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n\geq N} \quad |a_n - 0| < \varepsilon$$

$$\Longleftrightarrow \forall_{\varepsilon>0} \; \exists_{N\in\mathbb{N}} \; \forall_{n\geq N} \quad \big||a_n| - 0\big| < \varepsilon$$

$\Longleftrightarrow$ $(|a_n|)_{n\in\mathbb{N}}$ converges to zero.