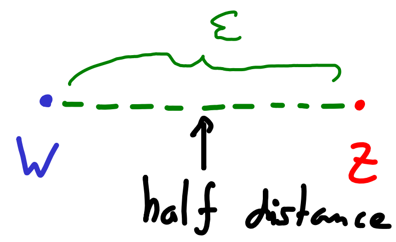
Problem 1 *Permutation of a sequence* (4 points)
Let $(z_n)_{n\in\mathbb{N}} \in \mathbb{C}$ be a convergent sequence with limit $z \in \mathbb{C}$ and $p : \mathbb{N} \to \mathbb{N}$ be bijective. Consider the sequence given by $w_n := z_{p(n)}$. Show that $(w_n)_{n\in\mathbb{N}}$ also converges to z.

Solutions

Claim: (w_n) converges to z

Proof: We know that for every $\varepsilon > 0$ only finitely many sequence members $\{z_n\}_{n \in I_\varepsilon}$, $|I_\varepsilon| < \infty$, fulfill: $|z_n - z| \geq \varepsilon$ $(*)$

Since $(z_n)_{n\in\mathbb{N}}$ is bounded (as a convergent sequence), $(w_n)_{n\in\mathbb{N}}$ is also bounded. By Bolzano-Weierstraß, we know that $(w_n)_{n\in\mathbb{N}}$ has at least one accumulation point w. Assume now $w \neq z$.



For $\varepsilon := \frac{1}{2}|z-w|$, we know that there are infinitely many sequence members $\{w_n\}_{n\in J_\varepsilon}$, $|J_\varepsilon| = \infty$, with $|w_n - w| < \varepsilon$.

Since $p: \mathbb{N} \to \mathbb{N}$ is bijective, also $p^{-1}(J_\varepsilon)$ is an infinite set. Therefore, for infinitely many members $\{z_k\}_{k \in p^{-1}(J_\varepsilon)}$, we have:

$$|z_k - w| < \varepsilon. \qquad \left(\text{substitution } k = p(n)\right)$$

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Hence: $|z - z_k| \geq |z - w| - |w - z_k|$

$$\geq 2\varepsilon - \varepsilon = \varepsilon \quad \text{for all } k \in p^{-1}(J_\varepsilon).$$

In contradiction to (*).