Problem 3 *Reordering an infinite sum* (4 points)
We consider the following "infinite sum", called a *series*,

$$\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\cdots$$

and denote the sum of the first n terms by S_n, also called the *nth partial sum*. For example, $S_4=\frac{1}{1}-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}$. By reordering, we get a new series:

$$\frac{1}{1}+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots.$$

We denote the nth partial sum of this new series by T_n. For example, $T_4=\frac{1}{1}+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}$. In this exercise, we examine if this reordering changes the limit of the partial sums. In order to do this, we use the *harmonic series* given by

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\cdots$$

and here we denote the nth partial sum by H_n.

a) Show that $H_{2n}-S_{2n}=H_n$ for all $n\in\mathbb{N}$.

b) Show that $T_{3n}+\frac{1}{2}H_n=H_{4n}-\frac{1}{2}H_{2n}$ for all $n\in\mathbb{N}$.

c) Assuming that $(S_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ are convergent, deduce

$$3\cdot\lim_{n\to\infty}S_n=2\cdot\lim_{n\to\infty}T_n\,.$$

Hint: Use the formula from (a) for S_{2n} and S_{4n} and the formula (b) for T_{3n}.

d) Explain in your own words, which property gets lost when one goes from a normal finite sum to an "infinite sum".

Solutions

(a) <u>Claim:</u> $H_{2n}-S_{2n}=H_n$ for all $n\in\mathbb{N}$

Proof: $H_{2n} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{2n}$

$S_{2n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{2n}$

Subtract!

$$H_{2n} - S_{2n} = \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n}\right) \cdot 2$$

$$= \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{n} = H_n \quad \checkmark$$

Of course the overall rigorous proof would be by induction:

base case: $n = 1$: $H_2 = \frac{1}{1} + \frac{1}{2}$, $S_2 = \frac{1}{1} - \frac{1}{2}$

$$\Longrightarrow H_2 - S_2 = \frac{1}{2} + \frac{1}{2} = 1 = H_1 \quad \checkmark$$

induction hypothesis: $H_{2n} - S_{2n} = H_n$ for a given $n \in \mathbb{N}$.

step case: $n \to n+1$

$$H_{2(n+1)} - S_{2(n+1)} = H_{2n+2} - S_{2n+2}$$

$$= H_{2n} + \frac{1}{2n+1} + \frac{1}{2n+2} - S_{2n} - \frac{1}{\underbrace{2n+1}_{\text{odd}}} + \frac{1}{\underbrace{2n+2}_{\text{even}}}$$

$$\overset{\text{i.h.}}{=} H_n + \frac{2}{2n+2} = H_n + \frac{1}{n+1} = H_{n+1} \quad \checkmark$$

(b)

Claim: $T_{3n} + \frac{1}{2} H_n = H_{4n} - \frac{1}{2} H_{2n}$ for all $n \in \mathbb{N}$.

Proof:
$$T_{3n} = \frac{1}{1} + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} \cdots + \frac{1}{4n-1} - \frac{1}{2n}$$
$$\frac{1}{2} H_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{2n-2} + \frac{1}{2n}$$

add

$$T_{3n} + \frac{1}{2} H_n = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{4n-1}$$

$$H_{4n} = \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{4n}$$
$$\frac{1}{2} H_{2n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots + \frac{1}{4n}$$

subtract:

$$H_{4n} - \frac{1}{2} H_{2n} = \frac{1}{1} + \frac{1}{3} + \cdots + \frac{1}{4n-1}$$

Of course the overall rigorous proof would be by induction:

base case: $n = 1$: $T_3 = \frac{1}{1} + \frac{1}{3} - \frac{1}{2}$, $\frac{1}{2} H_1 = \frac{1}{2}$
$-\frac{1}{2} H_2 = -\frac{1}{2} - \frac{1}{4}$

$$\Rightarrow T_3 + \frac{1}{2} H_1 = \frac{1}{1} + \frac{1}{3} = H_4 - \frac{1}{2} H_2$$

induction hypothesis: $T_{3n} + \frac{1}{2} H_n = H_{4n} - \frac{1}{2} H_{2n}$
for a given n.

step case: $n \to n+1$

$$T_{3(n+1)} + \frac{1}{2} H_{n+1} = T_{3n+3} + \frac{1}{2}\left(H_n + \frac{1}{n+1}\right)$$

$$= T_{3n} + \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{2(n+1)} + \frac{1}{2} H_n + \frac{1}{2n+2}$$

$$= T_{3n} + \frac{1}{2} H_{2n} + \frac{1}{4n+1} + \frac{1}{4n+3}$$

$$\overset{\text{i.h.}}{=} H_{4n} - \frac{1}{2} H_{2n} + \frac{1}{4n+1} + \frac{1}{4n+3}$$

$$= H_{4n+4} - \frac{1}{4n+2} - \frac{1}{4n+4} - \frac{1}{2}\left(H_{2n+2} - \frac{1}{2n+2} - \frac{1}{2n+1}\right)$$

$$= H_{4(n+1)} - \frac{1}{2} H_{2(n+1)} \quad \checkmark$$

(c) Claim: $3 \cdot \lim_{n\to\infty} S_n = 2 \cdot \lim_{n\to\infty} T_n$

Proof: By (a), we know:

$$S_{2n} = H_{2n} - H_n$$

$$2 \cdot S_{4n} = 2H_{4n} - 2H_{2n}$$

$$2 \cdot T_{3n} = 2 \cdot H_{4n} - H_{2n} - H_n$$

$$\Rightarrow \quad S_{2n} + 2S_{4n} = 2H_{4n} - H_{2n} - H_n$$

$$= 2T_{3n}$$

Since we know that $(S_n)_{n\in\mathbb{N}}$ and $(T_n)_{n\in\mathbb{N}}$ converge, also the subsequences $(S_{2n})_{n\in\mathbb{N}}$ and $(T_{3n})_{n\in\mathbb{N}}$ converge to the same limit, respectively. Also, the subsequence $(S_{4n})_{n\in\mathbb{N}}$ converges to $\lim\limits_{n\to\infty} S_n$.

Therefore, we get by the limit theorems:

$$3 \lim_{n\to\infty} S_n = 2 \cdot \lim_{n\to\infty} T_n \qquad \square$$

(d) Finite sums can be reordered by the associative and commutative law without changing the value of the sum. For "infinite sums" this reordering can indeed change the value of the sum.

The value of $S := \lim\limits_{n\to\infty} S_n$ is only $\frac{2}{3}$ times the value of $T := \lim\limits_{n\to\infty} T_n$.