Problem 3 *Sequences and subsequences* (4 points)
Determine all accumulation points (*also called limit points*) of the following sequences and conclude if the sequences converge.

a) $a_n := \sqrt{n+1} - \sqrt{n}$

b) $b_n := \begin{cases} n^2, & \text{for } n \text{ even} \\ \frac{1}{n^2}, & \text{for } n \text{ odd} \end{cases}$

c) $c_n := \left(1 - \frac{1}{n^2}\right)^n$

d) $d_n := \frac{2^n + (-3)^n}{(-2)^n + 3^n}$

Solutions

(a)
$$a_n = \sqrt{n+1} - \sqrt{n} = \frac{(\sqrt{n+1} - \sqrt{n}) \cdot (\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \frac{\frac{1}{\sqrt{n}} \xrightarrow{n\to\infty} 0}{\underbrace{\sqrt{1+\frac{1}{n}} + 1}_{\xrightarrow[n\to\infty]{} 2}} \xrightarrow[n\to\infty]{} 0$$

(see tutorial 2)

Convergent sequence has only one accumulation point, namely the limit itself and it is 0.

(b) Two subsequences: $b_{2n} = (2n)^2 \xrightarrow{n\to\infty} \infty$ unbounded

$$b_{2n-1} = \frac{1}{(2n-1)^2} \xrightarrow{n\to\infty} 0$$

Only one accumulation point: 0.
The sequence is unbounded and therefore not convergent

(c) $c_n = \Big(1 \underbrace{- \frac{1}{n^2}}_{h}\Big)^n \overset{\text{Bernoulli (see Sheet 0)}}{\geq} 1 + n\cdot\Big(-\frac{1}{n^2}\Big) = 1 - \frac{1}{n} \xrightarrow{n\to\infty} 1$

Since $1 \geq c_n \geq 1 - \frac{1}{n}$ Sandwich theorem is applicable.

Only one accumulation point: 1 and sequence is convergent!

(d) $d_n = \dfrac{\left(\frac{2}{3}\right)^n + (-1)^n}{\left(-\frac{2}{3}\right)^n + 1^n}$. Subsequences: $d_{2n} = \dfrac{q^{2n}+1}{q^{2n}+1}$ $\quad q = \frac{2}{3}$

$$d_{2n-1} = \frac{q^{2n-1}-1}{q^{2n-1}+1}$$

Since $|q|<1$ both converge with limit 1 and -1, respectively.

$\Rightarrow$ Two accumulation points $1, -1$ and (d_n) is not convergent.