Problem 4 *Monotonicity criterion for the infimum* (4 points)
Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ that is bounded from below and monotonically decreasing, which means that there is a number $c \in \mathbb{R}$ such that $c \leq a_n$ and $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. Prove that $(a_n)_{n\in\mathbb{N}}$ is a convergent sequence by using the completeness axiom (C) and the axiom of Archimedes (O5).

Completeness: If $(a_n)_{n\in\mathbb{N}}$ is a sequence in $\mathbb{R}$ such that for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ with $|a_n - a_m| < \varepsilon$ for all $n, m > N$, then the sequence converges.

Archimedes: For each $x \in \mathbb{R}$ and $\varepsilon > 0$, there is a natural number K with $x < K \cdot \varepsilon$.

Solutions

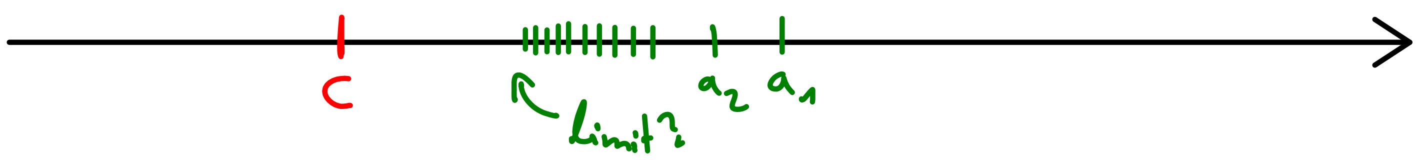
Claim: $(a_n)_{n\in\mathbb{N}}$ bounded from below and mon. decreasing $\Rightarrow$ $(a_n)_{n\in\mathbb{N}}$ converges

Proof: By the completeness axiom, we just have to show that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, which means:

$$\forall_{\varepsilon>0} \quad \exists_{N\in\mathbb{N}} \quad \forall_{n,m\geq N} \quad |a_n - a_m| < \varepsilon.$$

Now we know that there is a $c \in \mathbb{R}$ with $c \leq a_n$.

$(a_n - c \geq 0)$

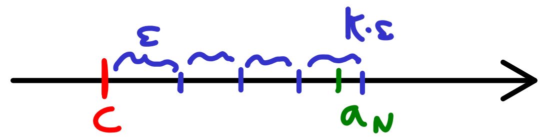


Let $\varepsilon > 0$ be arbitrary. Then we can find a number $K \in \mathbb{N}$ with the property

$$a_n - c \geq (K-1)\cdot\varepsilon \quad \text{for all } n \in \mathbb{N}$$

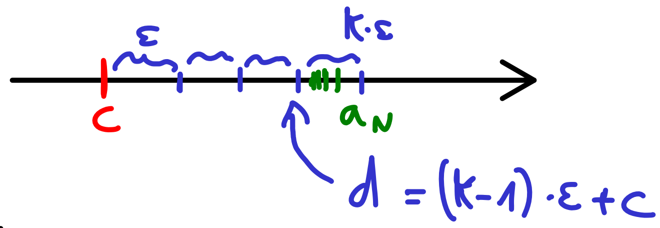
and $\quad a_N - c < K\cdot\varepsilon$ for some $N \in \mathbb{N}$. $\quad (*)$

This is possible because of the Archimedes axiom (O5).



Since $(a_n)_{n\in\mathbb{N}}$ is monot. decreasing, we know that the property (*) actually holds for all $n \geq N$:

$$a_n - c \leq a_N - c < k \cdot \varepsilon$$



We write $d := (k-1)\cdot\varepsilon + c$ and get:

$$d + \varepsilon = k\varepsilon + c > a_n \quad \text{for all } n \geq N. \qquad (a_n - d \geq 0)$$

Therefore for all $n, m \geq N$, we get:

$$|a_n - a_m| \leq |a_n - d| + |d - a_m| \qquad (\Delta\text{-inequality})$$

$$< \varepsilon + \varepsilon = 2\varepsilon \qquad (|a_n - d| = a_n - d < \varepsilon)$$

This shows that $(a_n)_{n\in\mathbb{N}}$ is a Cauchy sequence.
(The factor 2 can be absorbed by renaming ε with ε'.) $\square$