Problem 3 *Sequences* (4 points)
The following sequences are all convergent. Find their limits and justify that these numbers are indeed the limits of the particular sequences.

a) $a_n := \frac{n^2-n+4}{5n^3+n^2+3n}$.

b) $a_n := \frac{n}{n^2+1}$.

c) $a_n := \frac{2^n}{n!}$ where $n! := n(n-1)\cdots 2\cdot 1$.

d) $a_n := \frac{x^n}{n!}$ for a given $x \in \mathbb{R}$.

Solutions:

Problem 1.3

(a) $$a_n = \frac{n^2 - n - 4}{5n^3 + n^2 + 3n}$$ (limit should be 0)

$$= \frac{\frac{1}{n} - \frac{1}{n^2} - \frac{4}{n^3}}{5 + \frac{1}{n} - \frac{3}{n^2}} \qquad \begin{matrix} \xrightarrow{n\to\infty} 0 \\ \xrightarrow{n\to\infty} 5 \end{matrix}$$

Using Problem 1.2 (c), gives us the limit 0.

(b) $$a_n = \frac{n}{n^2+1}$$ (limit should be 0)

$$= \frac{\frac{1}{n}}{1 + \frac{1}{n^2}} \qquad \begin{matrix} \xrightarrow{n\to\infty} 0 \\ \xrightarrow{n\to\infty} 1 \end{matrix}$$

again 1.2 (c) $\Rightarrow$ limit $= 0$

(c) $a_n = \frac{2^n}{n!}$ (limit should be 0)

Set $N_0 > 4$. Let $\varepsilon > 0$ be arbitrary and choose $N > \max\left\{\frac{4}{\varepsilon}, N_0\right\}$.

Then for all $n \geq N$:

$$|a_n - 0| = \frac{2^n}{n!} = \frac{2}{n} \cdot \underbrace{\frac{2}{n-1}}_{<1} \cdots \underbrace{\frac{2}{N_0}}_{<1} \cdot \underbrace{\frac{2}{N_0-1} \cdots \frac{2}{2} \cdot \frac{2}{1}}_{\leq 2}$$

$$\leq \frac{2}{n} \cdot 2 \leq \frac{4}{N}$$

$$< \varepsilon$$

$\Longrightarrow$ limit is 0.

(d) $a_n = \frac{x^n}{n!}$ (limit should be 0)

Set $N_0 > |x|$. Let $\varepsilon > 0$ and choose $N > \max\left\{\frac{|x|^{N_0}}{\varepsilon}, N_0\right\}$.

Then for all $n \geq N$, we get:

$$|a_n - 0| = \frac{|x|^n}{n!} = \frac{|x|}{n} \frac{|x|}{n-1} \cdots \frac{|x|}{1}$$

$$= \frac{|x|}{n} \cdot \underbrace{\frac{|x|}{n-1}}_{<1} \cdots \underbrace{\frac{|x|}{N_0}}_{<1} \cdot \underbrace{\frac{|x|}{N_0-1} \cdots \frac{|x|}{1}}_{\leq |x|^{N_0-1}}$$

$$\leq \frac{|x|}{N} \cdot |x|^{N_0-1} = \frac{|x|^{N_0}}{N} < \varepsilon$$