Problem 1 *Elementary properties of real numbers* (4 points)
The set of real numbers is denoted by $\mathbb{R}$. Use the axioms of the real numbers given in the tutorial and prove the following properties:

a) $1 > 0$.

b) $\forall a, b \in \mathbb{R} : \big(a > b > 0 \;\Rightarrow\; b^{-1} > a^{-1}\big)$.

c) $\forall a, b \in \mathbb{R} : \big((a > b \wedge c < 0) \;\Rightarrow\; ca < cb\big)$.

d) $\forall a \in \mathbb{R} : \big(a \neq 0 \;\Rightarrow\; a^2 > 0\big)$.

Solutions

(a) <u>Claim:</u> $1 > 0$

<u>Proof:</u> From tutorial, we know:

$$0 \cdot x = 0 \quad , \quad -x = (-1)\,x \quad , \quad (-1)\cdot(-1) = 1$$

By (O1), we know that only one of the three claims is true: $\underbrace{0 = 1}_{\text{↯ (M2)}}$, $0 < 1$, $1 < 0$

If $1 < 0$ would be true, then $-1 > 0$ since

$$0 = 1 + (-1) \underset{(O3)}{<} 0 + (-1) = -1$$

However, then $1 = (-1)(-1) \underset{(O4)}{>} 0 \quad \Rightarrow \quad$ ↯ (O1)

(b) Claim: $a > b > 0 \Rightarrow \frac{1}{b} > \frac{1}{a}$

Proof: If $x > 0$, then $0 = x + (-x) \underset{(03)}{>} 0 + (-x) = -x$

which means $-x < 0$.

Therefore, if $x > 0$ then $x^{-1} > 0$ or $(-1)x^{-1} > 0$.

However, from the second possibility, we can conclude

$-1 = x(-1)x^{-1} > 0$ by (04) in contradiction to $1 > 0$.

Hence: $a > b \Rightarrow 1 \underset{(04)}{>} a^{-1}b \underset{(04)}{\Rightarrow} b^{-1} > a^{-1}$.

(c) Claim: $a > b > 0,\ c < 0 \Rightarrow ca < cb$

Proof: We know from before $c < 0 \Rightarrow (-1)c > 0$

Hence: $a > b \overset{(04)}{\Rightarrow} (-1)c \cdot a > (-1)c \cdot b$

$$\overset{(03)}{\Rightarrow} (-1)c \cdot a + cb > 0$$

$$\overset{(03)}{\Rightarrow} cb > ca$$

(d) Claim: $a \neq 0 \Rightarrow a^2 > 0$

Proof: If $a > 0 \overset{(O4)}{\Rightarrow} a \cdot a > 0 \Rightarrow a^2 > 0$ ✓

If $a < 0$, then $(-1) \cdot a > 0$.

$$\overset{(O4)}{\Rightarrow} (-1) a (-1) a > 0 \Rightarrow \underbrace{(-1)^2}_{=1} a^2 > 0$$

$$\Rightarrow a^2 > 0$$