Problem 2 *Supremum and Infimum* (4 points)
Examine if the following sets are bounded from above or from below and, when indicated, determine the supremum or infimum:

a) $A := \{(-1)^n + \frac{2}{n} \mid n \in \mathbb{N}\}$.

b) $B := \{(-1)^n - \frac{1}{n} \mid n \in \mathbb{N}\}$.

c) $C := \{1 + (-1)^n \cdot \frac{1}{n} \mid n \in \mathbb{N}\}$.

Solutions

(a) $A = \{(-1)^n + \frac{2}{n} \mid n \in \mathbb{N}\}$

Since $\left|(-1)^n + \frac{2}{n}\right| \leq |(-1)^n| + \left|\frac{2}{n}\right|,$

$$\leq 1 + 2 = 3$$

A is bounded (from below and above)

Claim: $\sup(A) = 2$

Proof: 2 is an upper bound since for $n = 1, 2$:

$$(-1)^1 + \frac{2}{1} = 1 \leq 2$$

$$(-1)^2 + \frac{2}{2} = 2 \leq 2 \quad (*)$$

and for $n \geq 3$ one has $\left|(-1)^n + \frac{2}{n}\right| \leq 1 + \underbrace{\frac{2}{n}}_{\leq 1} \leq 2.$

Because of $(*)$, 2 is also the smallest upper bound. $\square$

Claim: $\inf(A) = -1$

Proof: -1 is a lower bound since

$$-1 \leq -1 + \frac{2}{n} \leq (-1)^n + \frac{2}{n} \quad \text{for all } n \in \mathbb{N}.$$

For showing that -1 is the greatest lower bound, we consider an arbitrary $\varepsilon > 0$. Then choose $N \in \mathbb{N}$ odd with $N > \frac{2}{\varepsilon}$. Then $x := (-1)^N + \frac{2}{N} \in A$ fulfils $x < -1 + \varepsilon$ $\square$

(b) $B = \left\{ (-1)^n - \frac{1}{n} \mid n \in \mathbb{N} \right\}$

Since $\left|(-1)^n - \frac{1}{n}\right| \leq |(-1)^n| + \left|\frac{1}{n}\right| \leq 3$,

B is bounded.

Claim: $\sup(A) = 1$

Proof: 1 is upper bound: $(-1)^n - \frac{1}{n} \leq 1 - \frac{1}{n} \leq 1$.

Smallest upper bound: Choose for $\varepsilon > 0$ an even $N \in \mathbb{N}$ with $N > \frac{1}{\varepsilon}$. Then $x := (-1)^N - \frac{1}{N}$ fulfils

$$x > +1 - \varepsilon$$ $\square$

Claim: $\inf(A) = -2$

Proof: $-2 = -1 - 1 \leq (-1)^n - \frac{1}{n}$ $\rightarrow$ lower bound.

Greatest lower bound since $n = 1$: $(-1)^1 - \frac{1}{1} = -2$. $\square$

(c) $C = \left\{ 1 + (-1)^n \cdot \frac{1}{n} \;\middle|\; n \in \mathbb{N} \right\}$

Since $\left| 1 + (-1)^n \frac{1}{n} \right| \leq 1 + \frac{1}{n} \leq 2$,

C is bounded.

We have for $n = 1, 2$: $1 + (-1)^1 \cdot \frac{1}{1} = 0$

$1 + (-1)^2 \cdot \frac{1}{2} = \frac{3}{2}$

Obviously, $\inf C = 0$ and $\sup C = \frac{3}{2}$.