



Distributions - Part 4

Space of distributions

$\mathcal{D}(\mathbb{R}^n)$ - vector space of test functions
with notion of convergence for sequences of test functions
(special case of a topological vector space)

$T: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{R}$ (or \mathbb{C}) is called distribution if

- T is linear $\left(\begin{array}{l} T(\varphi_1 + \varphi_2) = T(\varphi_1) + T(\varphi_2) \\ T(\lambda\varphi) = \lambda \cdot T(\varphi) \end{array} \right)$

- T is continuous in the following sense:

For all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ and all sequences

$$(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n) \text{ with } \varphi_k \xrightarrow{\mathcal{D}} \varphi :$$

$$T(\varphi_k) \xrightarrow{k \rightarrow \infty} T(\varphi) \quad \left(\begin{array}{l} \text{sequentially} \\ \text{continuous} \end{array} \right)$$

Notation:

$\mathcal{D}(\mathbb{R}^n)'$ or $\mathcal{D}'(\mathbb{R}^n)$ space of distributions
important!

elements: $T: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{R}$

Examples: (a) delta distribution: $\delta: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{R}$

$$\delta(\varphi) = \varphi(0)$$

- linear ✓

- continuous: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n)$ with

$$\varphi_k \xrightarrow{\mathcal{D}} \varphi \quad \left(\begin{array}{l} \text{in particular:} \\ \varphi_k(x) \xrightarrow{k \rightarrow \infty} \varphi(x) \\ \text{for all } x \in \mathbb{R}^n \end{array} \right),$$

$$\text{we have: } \delta(\varphi_k) = \varphi_k(0) \xrightarrow{k \rightarrow \infty} \varphi(0) = \delta(\varphi) \quad \checkmark$$

(b) Continuous functions "are" distributions:

For $f \in C(\mathbb{R}^n)$ define: $T_f: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{R}$

$$T_f(\varphi) = \int_{\mathbb{R}^n} f(x) \varphi(x) dx$$

- linear ✓

- continuous: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n)$ with

$$\varphi_k \xrightarrow{\mathcal{D}} \varphi \quad \left(\begin{array}{l} \text{in particular:} \\ \varphi_k \xrightarrow{\text{uniform}} \varphi \\ \text{convergence} \end{array} \right),$$

$$\text{we have } T_f(\varphi_k) \xrightarrow{k \rightarrow \infty} T_f(\varphi) \quad \checkmark$$

Important property: $f, g \in C(\mathbb{R}^n)$, $f \neq g \Rightarrow T_f \neq T_g$

Beispiele: (a) delta distribution: $\delta: \mathcal{D}(\mathbb{R}^n) \rightarrow \mathbb{R}$

$$\delta(\varphi) = \varphi(0)$$

- linear ✓

- continuous: For $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $(\varphi_k)_{k \in \mathbb{N}} \subseteq \mathcal{D}(\mathbb{R}^n)$

$$\text{with } \varphi_k \xrightarrow{\mathcal{D}} \varphi \quad \left(\begin{array}{l} \text{in particular:} \\ \varphi_k(x) \xrightarrow{k \rightarrow \infty} \varphi(x) \\ \text{for all } x \in \mathbb{R}^n \end{array} \right),$$

$$\text{we have } \delta(\varphi_k) = \varphi_k(0) \xrightarrow{k \rightarrow \infty} \varphi(0) = \delta(\varphi) \quad \checkmark$$