

Unbounded Operators - Part 10

$T: X \supseteq \mathcal{D}(T) \rightarrow Y$ densely defined \implies adjoint exists:

$$(X, Y \text{ Banach spaces}) \quad T': Y' \supseteq \mathcal{D}(T') \rightarrow X'$$

$$(X, Y \text{ Hilbert spaces}) \quad T^*: Y \supseteq \mathcal{D}(T^*) \rightarrow X$$

Definition: Let $X = L^2(\mathbb{R}, \mathbb{C})$ ← square-integrable functions $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$
with respect to the
one-dimensional Lebesgue measure

Hilbert space with inner product:

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)} g(x) dx$$

Let $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ be a continuous function.

Then $M_\varphi: X \supseteq \mathcal{D}(M_\varphi) \rightarrow X$ denotes the multiplication operator:

$$f \mapsto M_\varphi f \quad \text{with} \quad (M_\varphi f)(x) = \varphi(x) f(x)$$

for $x \in \mathbb{R}$ almost everywhere

$$\mathcal{D}(M_\varphi) := \{ f \in L^2(\mathbb{R}, \mathbb{C}) \mid \varphi \cdot f \in L^2(\mathbb{R}, \mathbb{C}) \}$$

← dense in $L^2(\mathbb{R}, \mathbb{C})$

Adjoint of the multiplication operator: $(M_\varphi)^*: X \supseteq \mathcal{D}((M_\varphi)^*) \rightarrow X$

$$\{ g \in X \mid \text{there is } \tilde{f} \in X \text{ with } \langle g, M_\varphi f \rangle = \langle \tilde{f}, f \rangle \text{ for all } f \in \mathcal{D}(M_\varphi) \} \quad \equiv \quad \text{with} \quad (M_\varphi)^* g = \tilde{f}$$

Is it a multiplication operator as well?

$$\langle g, M_\psi f \rangle = \int_{\mathbb{R}} \overline{g(x)} \psi(x) f(x) dx = \int_{\mathbb{R}} \overline{\psi(x)} g(x) f(x) dx = \langle M_{\bar{\psi}} g, f \rangle$$

for all $f, g \in \mathcal{D}(M_\psi) = \mathcal{D}(M_{\bar{\psi}})$

First result: $M_{\bar{\psi}} \subseteq (M_\psi)^*$

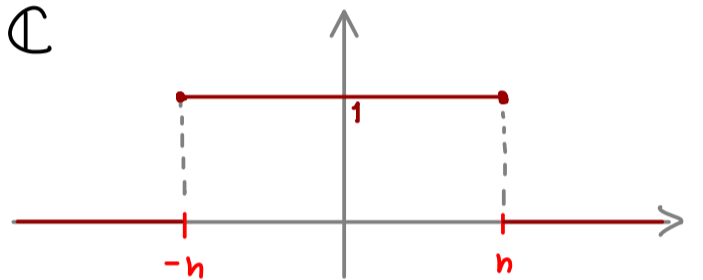
To show: $g \in \mathcal{D}((M_\psi)^*) \implies \bar{\psi} \cdot g \in L^2(\mathbb{R}, \mathbb{C})$

Proof: Note: $g \in L^2$, h bounded $\implies h \cdot g \in L^2$

Make $\bar{\psi}$ bounded? Take $\gamma_n: \mathbb{R} \rightarrow \mathbb{C}$

$\gamma_n \bar{\psi}$ is bounded

$$(\gamma_n \bar{\psi})(x) \xrightarrow{n \rightarrow \infty} \bar{\psi}(x) \text{ for } x \in \mathbb{R}$$



For $f \in \mathcal{D}(M_\psi)$, $g \in \mathcal{D}((M_\psi)^*)$:

$$\langle \gamma_n (M_\psi)^* g, f \rangle = \int_{\mathbb{R}} \overline{\gamma_n(x) (M_\psi)^* g(x)} f(x) dx$$

$$= \langle (M_\psi)^* g, \gamma_n f \rangle = \langle g, M_\psi (\gamma_n f) \rangle$$

$$= \int_{\mathbb{R}} \overline{g(x)} \psi(x) \gamma_n(x) f(x) dx$$

$$= \int_{\mathbb{R}} \overline{\psi(x) \gamma_n(x) g(x)} f(x) dx = \langle \gamma_n \bar{\psi} g, f \rangle$$

$\mathcal{D}(M_\psi)$ dense

$$\implies \gamma_n (M_\psi)^* g = \gamma_n \bar{\psi} g \xrightarrow{\gamma_n \xrightarrow{n \rightarrow \infty} 1} (M_\psi)^* g = \bar{\psi} g \in L^2$$

Final result: $(M_\psi)^* = M_{\bar{\psi}}$