The Bright Side of Mathematics

Unbounded Operators – Part 5

$T: X \supseteq \mathcal{D}(T) \longrightarrow Y$ closable $\Longleftrightarrow$ For each $(x_n) \subseteq \mathcal{D}(T)$ with $x_n \to 0$ and $Tx_n \longrightarrow y$, we have: $y = 0$.

Example: $X = \ell^2(\mathbb{N}, \mathbb{C})$, $e_1, e_2, e_3, \dots$ canonical unit vectors ($e_2 = (0,1,0,0,\dots)$)

$$T: X \supseteq \mathcal{D}(T) \longrightarrow \mathbb{C}, \quad \mathcal{D}(T) = \operatorname{span}\{e_j \mid j \in \mathbb{N}\}$$

$$e_j \longmapsto j$$

$$\sum_j \lambda_j e_j \longmapsto \sum_j \lambda_j \cdot j$$

$$\|T\| = \sup_{\|x\|_X = 1} \|Tx\|_{\mathbb{C}} \geq \sup_{j \in \mathbb{N}} |Te_j| = \sup_{j \in \mathbb{N}} j = \infty$$

unbounded operator! → not continuous at 0

Closable operator?

Choose $(x_n) \subseteq \mathcal{D}(T)$ with $x_n \to 0$ and $Tx_n \not\to 0$.

Choose $\varepsilon > 0$ and subsequence (x_{n_k}) such that: $|Tx_{n_k}| \geq \varepsilon$

Define: $z_k := \dfrac{x_{n_k}}{Tx_{n_k}} \xrightarrow{k \to \infty} 0$

Then: $Tz_k = 1$ for all $k \in \mathbb{N}$ ($1 = y$)

$\Longrightarrow$ T is not closable

For each $(x_n) \subseteq \mathcal{D}(T)$ with $x_n \to 0$ and $Tx_n \longrightarrow y$, we have: $y = 0$.