Unbounded Operators – Part 6

Closed Graph Theorem: X, Y Banach spaces, $T: X \supseteq D(T) \to Y$ operator with $D(T)$ closed (e.g. $D(T) = X$).

Then: T closed $\Longleftrightarrow$ T continuous (bounded)

Proof: Assume: $D(T) = X$.

($\Leftarrow$) Choose $(x_n) \subseteq D(T)$ with $x_n \to x \in X$ and $Tx_n \to y \in Y$

$$\overset{T \text{ continuous}}{\Longrightarrow} \quad y = \lim_{n\to\infty} T(x_n) = T\left(\lim_{n\to\infty} x_n\right) = Tx$$

$$\Longrightarrow \quad x \in D(T) \text{ and } Tx = y \quad \Longrightarrow \quad T \text{ closed}$$

($\Rightarrow$) Assume T is closed $\Longrightarrow$ G_T is closed in $X \times Y$ $\Longrightarrow$ $(G_T, \|\cdot\|_{X\times Y})$ Banach space

Define operators: $P_X : G_T \to X$, $(x,y) \mapsto x$ and $P_Y : G_T \to Y$, $(x,y) \mapsto y$ linear + bounded

P_X: bijective!

Bounded Inverse Theorem

Functional Analysis – Part 27

$$\Longrightarrow \quad P_X^{-1} : X \to G_T, \quad x \mapsto (x, Tx) \quad \text{is continuous (bounded operator)}$$

$$\begin{tikzcd}
X \arrow[rr, "T"] \arrow[dr, "P_X^{-1}"'] & & Y \\
& G_T \arrow[ur, "P_Y"'] &
\end{tikzcd}$$

$T = P_Y P_X^{-1}$ composition of continuous maps

$\Longrightarrow$ T continuous (bounded)