# Unbounded Operators — Part 6

**Closed Graph Theorem:** $X, Y$ Banach spaces, $T : X \supseteq D(T) \to Y$ operator with $D(T)$ closed $\left(\text{e.g. } D(T) = X\right)$.

Then: $T$ closed $\iff$ $T$ continuous (bounded)

**Proof:** Assume: $D(T) = X$.

$(\Leftarrow)$ Choose $(x_n) \subseteq D(T)$ with $x_n \to x \in X$ and $Tx_n \to y \in Y$

$$\overset{T \text{ continuous}}{\Longrightarrow} \quad y = \lim_{n\to\infty} T(x_n) = T\left(\lim_{n\to\infty} x_n\right) = Tx$$

$$\Longrightarrow \quad x \in D(T) \text{ and } Tx = y \quad \Longrightarrow \quad T \text{ closed}$$

$(\Rightarrow)$ Assume $T$ is closed $\Rightarrow$ $G_T$ is closed in $X \times Y$ $\Rightarrow$ $\left(G_T, \|\cdot\|_{X\times Y}\right)$ Banach space

Define operators: $P_X : G_T \to X$, $(x,y) \mapsto x$ and $P_Y : G_T \to Y$, $(x,y) \mapsto y$ linear + bounded

$P_X$: bijective!

Bounded Inverse Theorem

Functional Analysis – Part 27

$$\Longrightarrow \quad P_X^{-1} : X \to G_T \text{ is continuous (bounded operator)}$$
$$x \mapsto (x, Tx)$$

$$\begin{tikzcd} X \arrow[rr, "T"] \arrow[dr, "P_X^{-1}"'] & & Y \\ & G_T \arrow[ur, "P_Y"'] & \end{tikzcd}$$

$T = P_Y P_X^{-1}$ composition of continuous maps

$\Longrightarrow$ $T$ continuous (bounded)