

Unbounded Operators - Part 6

Closed Graph Theorem: X, Y Banach spaces, $T: X \supseteq \mathcal{D}(T) \rightarrow Y$ operator
with $\mathcal{D}(T)$ closed (e.g. $\mathcal{D}(T) = X$).

Then: T closed $\Leftrightarrow T$ continuous (bounded)

Proof: Assume: $\mathcal{D}(T) = X$.

(\Leftarrow) Choose $(x_n) \subseteq \mathcal{D}(T)$ with $x_n \rightarrow x \in X$ and $Tx_n \rightarrow y \in Y$

$$\stackrel{T \text{ continuous}}{\Rightarrow} y = \lim_{n \rightarrow \infty} T(x_n) = T\left(\lim_{n \rightarrow \infty} x_n\right) = Tx$$

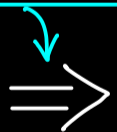
$$\Rightarrow x \in \mathcal{D}(T) \text{ and } Tx = y \Rightarrow T \text{ closed}$$

(\Rightarrow) Assume T is closed $\Rightarrow G_T$ is closed in $X \times Y \Rightarrow (G_T, \|\cdot\|_{X \times Y})$ Banach space

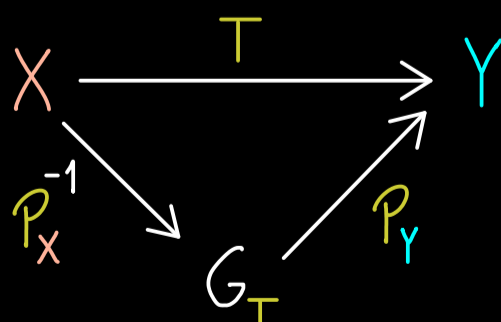
Define operators: $P_X: G_T \rightarrow X$ and $P_Y: G_T \rightarrow Y$ linear + bounded
 $(x, y) \mapsto x$ $(x, y) \mapsto y$
 bijective!

Bounded Inverse Theorem

Functional Analysis - Part 27



$P_X^{-1}: X \rightarrow G_T$ is continuous (bounded operator)
 $x \mapsto (x, Tx)$



$T = P_Y P_X^{-1}$ composition of continuous maps

$\Rightarrow T$ continuous (bounded)