Unbounded Operators - Part 10

$T: X \supseteq D(T) \to Y$ densely defined $\Longrightarrow$ adjoint exists:

(X, Y Banach spaces) $T': Y' \supseteq D(T') \to X'$

(X, Y Hilbert spaces) $T^*: Y \supseteq D(T^*) \to X$

Definition: Let $X = L^2(\mathbb{R}, \mathbb{C})$ ← square-integrable functions $\int_{\mathbb{R}} |f(x)|^2 dx < \infty$ with respect to the one-dimensional Lebesgue measure

Hilbert space with inner product:

$$\langle f, g \rangle = \int_{\mathbb{R}} \overline{f(x)}\, g(x)\, dx$$

Let $\varphi: \mathbb{R} \to \mathbb{C}$ be a continuous function.

Then $M_\varphi: X \supseteq D(M_\varphi) \to X$ denotes the multiplication operator:

$f \mapsto M_\varphi f$ with $(M_\varphi f)(x) = \varphi(x) f(x)$

for $x \in \mathbb{R}$ almost everywhere

$$D(M_\varphi) := \{ f \in L^2(\mathbb{R}, \mathbb{C}) \mid \varphi \cdot f \in L^2(\mathbb{R}, \mathbb{C}) \}$$

dense in $L^2(\mathbb{R}, \mathbb{C})$

Adjoint of the multiplication operator: $(M_\varphi)^*: X \supseteq D((M_\varphi)^*) \to X$

$\{ g \in X \mid \text{there is } \tilde{f} \in X \text{ with } \langle g, M_\varphi f \rangle = \langle \tilde{f}, f \rangle \text{ for all } f \in D(M_\varphi) \}$ // with $(M_\varphi)^* g = \tilde{f}$

Is it a multiplication operator as well?

$$\langle g, M_\varphi f \rangle = \int_{\mathbb{R}} \overline{g(x)} \varphi(x) f(x)\, dx = \int_{\mathbb{R}} \overline{\overline{\varphi(x)}\, g(x)}\, f(x)\, dx = \langle M_{\overline{\varphi}}\, g, f \rangle$$

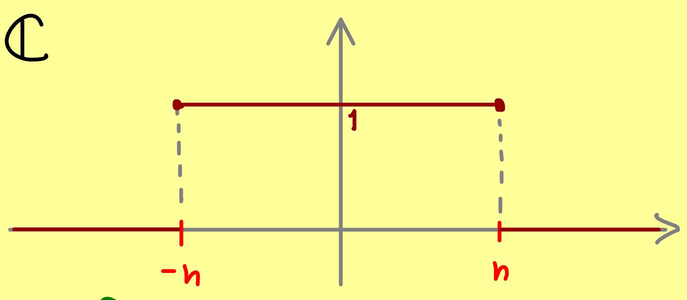
for all $f, g \in D(M_\varphi) = D(M_{\overline{\varphi}})$

First result: $M_{\overline{\varphi}} \subseteq (M_\varphi)^*$

To show: $g \in D((M_\varphi)^*) \Longrightarrow \overline{\varphi} \cdot g \in L^2(\mathbb{R}, \mathbb{C})$

Proof: Note: $g \in L^2$, h bounded $\Longrightarrow$ $h \cdot g \in L^2$

Make $\overline{\varphi}$ bounded? Take $\gamma_n: \mathbb{R} \to \mathbb{C}$



$\gamma_n \overline{\varphi}$ is bounded

$(\gamma_n \overline{\varphi})(x) \xrightarrow{n \to \infty} \overline{\varphi}(x)$ for $x \in \mathbb{R}$

For $f \in D(M_\varphi)$, $g \in D((M_\varphi)^*)$:

$$\langle \gamma_n (M_\varphi)^* g, f \rangle = \int_{\mathbb{R}} \overline{\gamma_n(x) (M_\varphi)^* g(x)}\, f(x)\, dx$$

$$= \langle (M_\varphi)^* g, \gamma_n f \rangle = \langle g, M_\varphi(\gamma_n f) \rangle$$

$$= \int_{\mathbb{R}} \overline{g(x)}\, \varphi(x) \gamma_n(x) f(x)\, dx$$

$$= \int_{\mathbb{R}} \overline{\overline{\varphi(x)} \gamma_n(x)\, g(x)}\, f(x)\, dx = \langle \gamma_n \overline{\varphi}\, g, f \rangle$$

$\overset{D(M_\varphi) \text{ dense}}{\Longrightarrow}$ $\gamma_n (M_\varphi)^* g = \gamma_n \overline{\varphi}\, g$ $\overset{\gamma_n \xrightarrow{n\to\infty} 1}{\Longrightarrow}$ $(M_\varphi)^* g = \overline{\varphi}\, g \in L^2$

Final result: $(M_\varphi)^* = M_{\overline{\varphi}}$