



Unbounded Operators - Part 9

X, Y Banach spaces

$$T: X \supseteq \mathcal{D}(T) \rightarrow Y$$

densely defined operator

$$\Rightarrow T': Y' \supseteq \mathcal{D}(T') \rightarrow X'$$

(Banach space) adjoint operator

X, Y Hilbert spaces

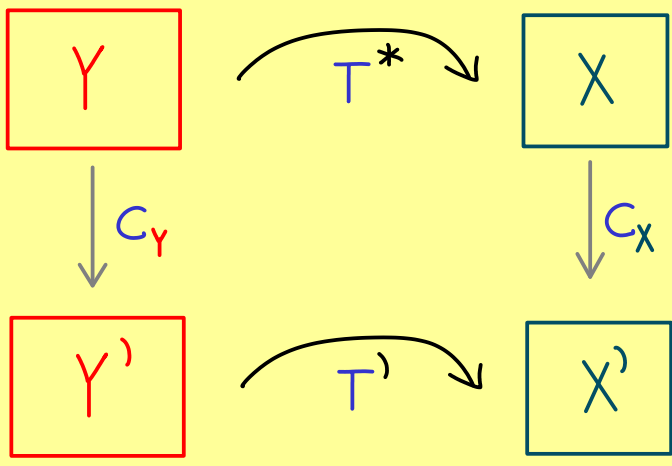
$$T: X \supseteq \mathcal{D}(T) \rightarrow Y$$

densely defined operator

$$\Rightarrow T^*: Y \supseteq \mathcal{D}(T^*) \rightarrow X$$

(Hilbert space) adjoint operator

Connection between T' and T^* :



Riesz representation theorem: $X' \cong X$ (for Hilbert spaces)

$$C_X: X \rightarrow X', x \mapsto \langle x, \cdot \rangle_X = \langle x |$$

$$C_Y: Y \rightarrow Y', y \mapsto \langle y, \cdot \rangle_Y = \langle y |$$

antilinear isometric isomorphism

where $T'(\langle y |)(x) = \langle y, Tx \rangle_Y$
for $x \in \mathcal{D}(T) = \langle T^*y, x \rangle_Y$

$$C_X^{-1} T' C_Y(y) = C_X^{-1} T'(\langle y |) \text{ for } y \in \mathcal{D}(T^*)$$

$$= C_X^{-1}(\langle T^*y |)$$

$$= T^*y$$

$$\Rightarrow T^* = C_X^{-1} T' C_Y$$

Proposition: X, Y Banach spaces, $T: X \supseteq \mathcal{D}(T) \rightarrow Y$ densely defined operator.

Then: $T \subseteq S \Rightarrow T' \supseteq S'$

$$\left(\begin{array}{l} \mathcal{D}(T) \subseteq \mathcal{D}(S), S \text{ extension of } T \\ Sx = Tx \text{ for all } x \in \mathcal{D}(T) \end{array} \right) \left(\begin{array}{l} \mathcal{D}(T') \supseteq \mathcal{D}(S'), S' \text{ restriction of } T' \\ S'y = T'y \text{ for all } y \in \mathcal{D}(S') \end{array} \right)$$

And for Hilbert spaces: $T \subseteq S \Rightarrow T^* \supseteq S^*$

Proof: $\mathcal{D}(S') := \{ y' \in Y' \mid \text{there is } x' \in X' \text{ with } y'(Sx) = x'(x) \text{ for all } x \in \mathcal{D}(S) \}$

$$\subseteq \{ y' \in Y' \mid \text{there is } x' \in X' \text{ with } y'(Tx) = x'(x) \text{ for all } x \in \mathcal{D}(T) \}$$

$$= \mathcal{D}(T') \quad \square$$