The Bright Side of Mathematics

Unbounded Operators - Part 9

X, Y Banach spaces	X, Y Hilbert spaces
$T: X \supseteq \mathcal{D}(T) \to Y$	$T: X \supseteq \mathcal{D}(T) \to Y$
densely defined operator	densely defined operator
$\Longrightarrow \; T': Y' \supseteq \mathcal{D}(T') \to X'$	$\Longrightarrow \; T^*: Y \supseteq \mathcal{D}(T^*) \to X$
(Banach space) adjoint operator	(Hilbert space) adjoint operator

Connection between T' and T^*:

$$\begin{tikzcd} \boxed{Y} \arrow[r, bend left, "T^*"] \arrow[d, "C_Y"] & \boxed{X} \arrow[d, "C_X"] \\ \boxed{Y'} \arrow[r, bend left, "T'"] & \boxed{X'} \end{tikzcd}$$

Riesz representation theorem: $X' \cong X$ antilinear isometric isomorphism
(for Hilbert spaces)

$$C_X: X \to X', \quad x \mapsto \langle x, \cdot \rangle_X = \langle x|$$

$$C_Y: Y \to Y', \quad y \mapsto \langle y, \cdot \rangle_Y = \langle y|$$

where $T'(\langle y|)(x) = \langle y, Tx \rangle_Y$
for $x \in \mathcal{D}(T)$ $\quad = \langle T^* y, x \rangle_Y$

$$C_X^{-1} T' C_Y(y) = C_X^{-1} T'(\langle y|) \quad \text{for } y \in \mathcal{D}(T^*)$$

$$= C_X^{-1}(\langle T^* y|)$$

$$= T^* y$$

$$\Longrightarrow \quad T^* = C_X^{-1} T' C_Y$$

<u>Proposition:</u> X, Y Banach spaces, $T: X \supseteq \mathcal{D}(T) \to Y$ densely defined operator.

Then: $T \subseteq S \quad \Longrightarrow \quad T' \supseteq S'$

$\left(\mathcal{D}(T) \subseteq \mathcal{D}(S), \; S \text{ extension of } T, \; Sx = Tx \text{ for all } x \in \mathcal{D}(T) \right)$ $\quad \left(\mathcal{D}(T') \supseteq \mathcal{D}(S'), \; S' \text{ restriction of } T', \; S'y' = T'y' \text{ for all } y' \in \mathcal{D}(S') \right)$

And for Hilbert spaces: $T \subseteq S \quad \Longrightarrow \quad T^* \supseteq S^*$

<u>Proof:</u> $\mathcal{D}(S') := \left\{ y' \in Y' \;\middle|\; \text{there is } x' \in X' \text{ with } y'(Sx) = x'(x) \text{ for all } x \in \mathcal{D}(S) \right\}$

$$\subseteq \left\{ y' \in Y' \;\middle|\; \text{there is } x' \in X' \text{ with } y'(Tx) = x'(x) \text{ for all } x \in \mathcal{D}(T) \right\}$$

$$= \mathcal{D}(T') \qquad \square$$