Unbounded Operators – Part 8

For bounded operators: $T: X \to Y \rightsquigarrow T^*: Y \to X$ adjoint

Hilbert spaces

$$\langle y, Tx \rangle_Y = \langle T^* y, x \rangle_X$$

$T: X \to Y \rightsquigarrow T': Y' \to X'$ adjoint

Banach spaces

$$T'(y')(x) = y'(Tx)$$

for $y' \in Y'$, $x \in X$

Proposition: X, Y Banach spaces, $T: X \supseteq D(T) \to Y$ densely defined operator

$\hookrightarrow \overline{D(T)} = X$

Then there is an operator $T': Y' \supseteq D(T') \to X'$ with

$$y'(Tx) = T'(y')(x) \quad \text{for } x \in D(T),\ y' \in D(T').$$

The domain $D(T')$ can be chosen maximally.

Proof: set $D(T') := \{ y' \in Y' \mid \text{there is } x' \in X' \text{ with } y'(Tx) = x'(x) \text{ for all } x \in D(T) \}$

and define: $T'(y') := x'$

Well-defined? Assume there are $x_1', x_2' \in X'$ with

$$\left.\begin{aligned} y'(Tx) &= x_1'(x) \\ y'(Tx) &= x_2'(x) \end{aligned}\right\} \text{for all } x \in D(T)$$

$\Longrightarrow x_1'(x) = x_2'(x)$ for all $x \in D(T)$

$\Longrightarrow (x_1' - x_2')(x) = 0$ for all $x \in D(T)$ $\xRightarrow[\text{continuity}]{\text{dense}}$ $(x_1' - x_2')(x) = 0$ for all $x \in X$

$\Longrightarrow x_1' = x_2'$ $\square$

For Hilbert spaces: X, Y Hilbert spaces, $T: X \supseteq D(T) \to Y$ densely defined operator

$\hookrightarrow \overline{D(T)} = X$

$$D(T^*) := \left\{ y \in Y \;\middle|\; \text{there is } \tilde{x} \in X \text{ with } \langle y, Tx \rangle_Y = \langle \tilde{x}, x \rangle_X \text{ for all } x \in D(T) \right\}$$

$$T^*(y) := \tilde{x}$$