Unbounded Operators – Part 7

X, Y Banach spaces, $T : X \supseteq D(T) \to Y$ operator

Closed Graph Theorem: $D(T) = X \implies \left(T \text{ closed} \iff T \text{ bounded} \right)$

Example: functional $T : X \to \mathbb{C}$ unbounded (see part 5)

extend: $D(T) = X$

$\implies T$ not closed

Proposition: X, Y Banach spaces, $T : X \supseteq D(T) \to Y$ operator.

Then: T closed $\iff (D(T), \|\cdot\|_T)$ complete

graph norm

$\|x\|_T := \|x\|_X + \|Tx\|_Y$

Proof: $J : (D(T), \|\cdot\|_T) \longrightarrow (G_T, \|\cdot\|_{X \times Y})$, $x \longmapsto (x, Tx)$ linear + bijective

$$\|Jx\|_{X \times Y} = \|(x, Tx)\|_{X \times Y} = \|x\|_X + \|Tx\|_Y = \|x\|_T$$

$\implies J$ is an isometric isomorphism

$(D(T), \|\cdot\|_T)$ complete $\iff (G_T, \|\cdot\|_{X \times Y})$ complete

$\iff (G_T, \|\cdot\|_{X \times Y})$ closed in $X \times Y$

$\iff T$ closed