Unbounded Operators – Part 2

Recall: operator $T: X \to Y$ with $\mathcal{D}(T) = \mathcal{D}$

means: $T: \mathcal{D} \to Y$ linear map

Fact: If $\mathrm{Ker}(T) = \{0\}$, then $T^{-1}: Y \to X$ with $\mathcal{D}(T^{-1}) = \mathrm{Ran}(T)$

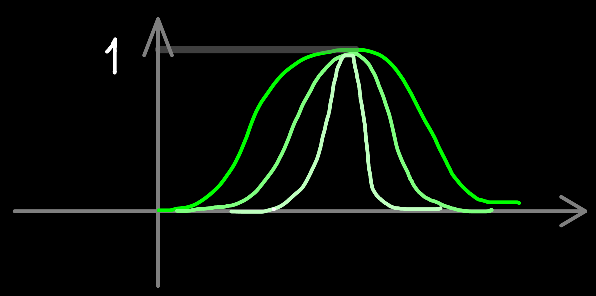
↳ always defined as an operator

Examples: $X = Y = C([0,1])$ (with supremum norm $\|\cdot\|_\infty$)

(a) $T: X \to Y$ with $\mathcal{D}(T) = C^1([0,1])$

$Tx = x'$

unbounded operator



$$\|T\| = \sup_{\|x\|_\infty = 1} \|Tx\|_\infty = \sup_{\|x\|_\infty = 1} \|x'\|_\infty = \infty$$

(b) $S: X \to Y$ with $\mathcal{D}(S) = \{ x \in C^1([0,1]) \mid x(0) = 0 \}$

$Sx = x'$

notations: $S \subseteq T$

the operator T is an extension of S

the operator S is a restriction of T

Note:

- $\mathrm{Ker}(T) \neq \{0\}$ not injective!
- $\mathrm{Ker}(S) = \{0\}$ injective! $\Rightarrow$ S^{-1} exists
- T is densely defined $\left(\overline{C^1([0,1])}^{\|\cdot\|_\infty} = C([0,1]) \right)$
- S is not densely defined