Unbounded Operators – Part 1

Linear Algebra Real Analysis

Functional Analysis

Motivation:

- partial differential equations
- quantum mechanics: one needs operators X, P with

$$XP - PX = i \cdot I$$

Definition: Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ be normed spaces (same field $\mathbb{F} \in \{\mathbb{R}, \mathbb{C}\}$)

and $\mathcal{D} \subseteq X$ subspace.

A linear map $T : \mathcal{D} \longrightarrow Y$ is called an operator.

Other notations:
- $T : X \supseteq \mathcal{D} \longrightarrow Y$
- $T : X \longrightarrow Y$ with domain $\mathcal{D}$
- $(T, \mathcal{D})$ or T with $\mathcal{D}(T) = \mathcal{D}$

Moreover: T is called densely defined if $\overline{\mathcal{D}}^{\|\cdot\|_X} = X$.

$$\mathrm{Ran}(T) := \{ Tx \mid x \in \mathcal{D} \} \subseteq Y \quad \text{subspace}$$

$$\mathrm{Ker}(T) := \{ x \in \mathcal{D} \mid Tx = 0 \} \subseteq X \quad \text{subspace}$$

T is called bounded if $\exists C > 0 \ \forall x \in \mathcal{D} : \|Tx\|_Y \leq C \cdot \|x\|_X$

T is called unbounded if $\forall C > 0 \ \exists x \in \mathcal{D} : \|Tx\|_Y > C \cdot \|x\|_X$

Recall: T is bounded $\Longleftrightarrow$ T is continuous at all points $x \in \mathcal{D}$

Therefore: T is unbounded $\Longleftrightarrow$ T is not continuous (at no point $x \in \mathcal{D}$)