The Bright Side of Mathematics

The following pages cover the whole Start Learning Sets course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

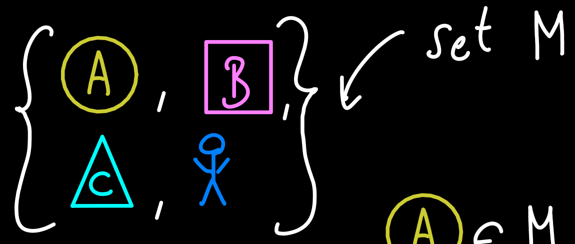
Have fun learning mathematics!

Start Learning Sets – Part 1

Propositional Logic + Naive set theory $\rightsquigarrow$ **Goal:** doing mathematics

Logic + Axioms of set theory $\rightsquigarrow$ foundation of mathematics $\rightsquigarrow$ doing mathematics

Set: Collection of distinct objects into a whole



Such an object x inside a set M is called an element of M, write: $x \in M$.

$A \in M$

If x is not such an object inside the set M, we write: $x \notin M$ means: $\neg(x \in M)$

$D \notin M$

A set can be defined by giving all its elements: $A \underset{\text{defined by}}{:=} \{2, 5, 6\}$

Examples: Empty set: $\emptyset := \{\}$

Natural numbers: $\mathbb{N} := \{1, 2, 3, 4, 5, \dots\}$

Natural numbers (including zero): $\mathbb{N}_0 := \{0, 1, 2, 3, 4, \dots\}$

Integers: $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$

Rational numbers $\mathbb{Q}$ Real numbers $\mathbb{R}$ Complex numbers $\mathbb{C}$

$\rightsquigarrow$ quantifiers $\forall\ \exists$ predicates $x \in \mathbb{N}$

Start Learning Sets - Part 2

$\boxed{1}$ is an even number — false logical statement

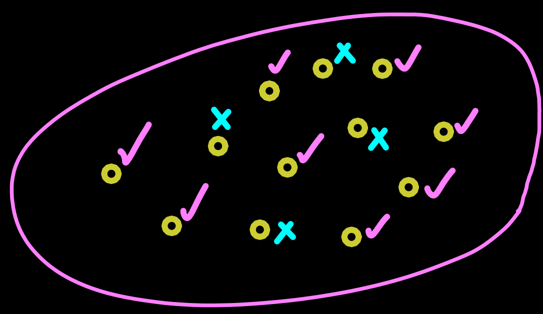
$\boxed{1}$ is an animal — false logical statement

$\underbrace{\boxed{1} + 8 = 9}_{\text{predicates}}$ — true logical statement

Predicate: An expression with undetermined variables that ascribes a property to objects filled in for the variables.

Form new sets:

$$\{ x \in \mathbb{N} \mid x \text{ is an even number} \}$$



$\mathbb{N}$

$$\{ y \in \mathbb{Z} \mid y \in \mathbb{N} \}$$

For $A := \{ \text{Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune} \}$

form: $\{ p \in A \mid p \text{ has at least 1 confirmed moon} \}$

Quantifiers: $\forall x$ for all x $\qquad$ $\exists x$ it exists x

Predicate: x is a planet

$\underbrace{\forall x : x \text{ is a planet}}_{\text{false}} \rightsquigarrow$ logical statement

$\underbrace{\exists x : x \text{ is a planet}}_{\text{true}} \rightsquigarrow$ logical statement

<u>Equality for sets:</u> Two sets A, B are the same, written as $A = B$ if

$$\forall x : \quad x \in A \leftrightarrow x \in B \qquad \text{is true.}$$

<u>Example:</u> $C := \{2, 3, 5\} = \{3, 5, 2\} =: D$ $\qquad 1 \in C \leftrightarrow 1 \in D$ true

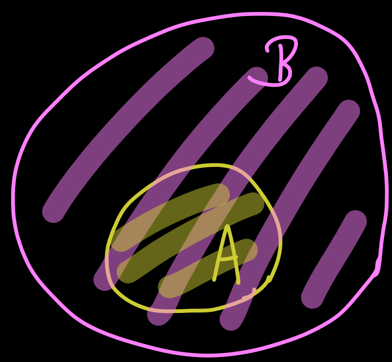
$2 \in C \leftrightarrow 2 \in D$ true

$\vdots$

$$\{2, 3, 5\} = \{2, 2, 2, 3, 3, 5\}$$

<u>Subsets:</u> For two sets A, B, we write $A \subseteq B$ if

$$\underbrace{\forall x : \quad x \in A \rightarrow x \in B}_{\text{Short notation: } \forall x \in A : \ x \in B} \qquad \text{is true.}$$



We call <u>A a subset of B.</u> (We can also write $B \supseteq A$)

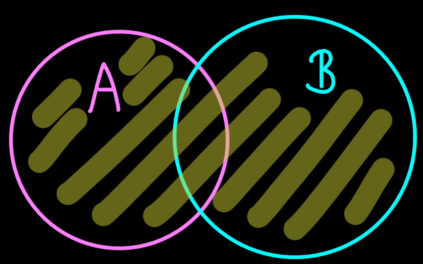
Start Learning Sets – Part 3

$A \subseteq B$ ← is a superset of A
↖ is a subset of B

$\rightsquigarrow$ $B \subseteq B$ ✓
$\emptyset \subseteq B$ ✓

$\forall x : x \in \emptyset \rightarrow x \in B$

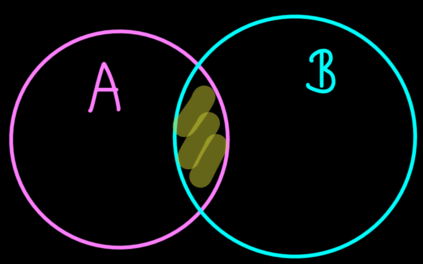
Union:



$$A \cup B := \{ x \mid x \in A \lor x \in B \}$$

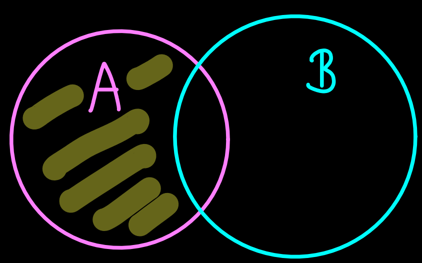
$(\forall x : \; x \in A \cup B \leftrightarrow x \in A \lor x \in B)$ is true

Intersection:



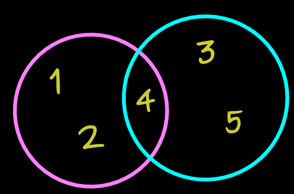
$$A \cap B := \{ x \mid x \in A \land x \in B \}$$

Set difference:



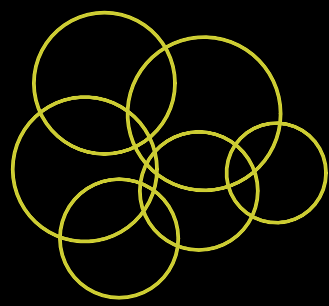
$$A \setminus B := \{ x \mid x \in A \land x \notin B \}$$

Example: $A := \{1, 2, 4\}$, $B := \{3, 4, 5\}$



$A \cup B = \{1, 2, 3, 4, 5\}$, $A \cap B = \{4\}$, $A \setminus B = \{1, 2\}$

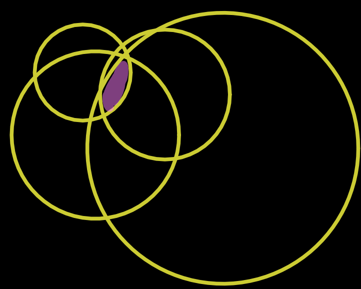
Big union: Need: I set, A_i set for each $i \in I$.



$$\bigcup_{i \in I} A_i := \{ x \mid \exists i \in I : x \in A_i \}$$

Big intersection:

$$\bigcap_{i \in I} A_i := \{ x \mid \forall i \in I : x \in A_i \}$$



Example: $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$, ...

$I = \mathbb{N}$, $A_i = \{i\}$. Then: $\bigcup_{i \in I} A_i = \{1, 2, 3, \dots\} = \mathbb{N}$

$$\bigcap_{i \in I} A_i = \emptyset$$

Power set: For a set A define $\mathcal{P}(A) := \{ X \mid X \subseteq A \}$ The set of all subsets of A

Example: $A = \{1, 2, 3\}$, $\mathcal{P}(A) = \{ \emptyset, \{1,2,3\}, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\} \}$

Number of elements: $|A| = 3$, $|\mathcal{P}(A)| = 8 = 2^3$

Start Learning Sets – Part 4

Cartesian product: $A \times B$ set of all ordered pairs

$A := \{\triangle, \square, \circ\}$ $\rightsquigarrow (\triangle, 7)$

$B := \{4, 7\}$

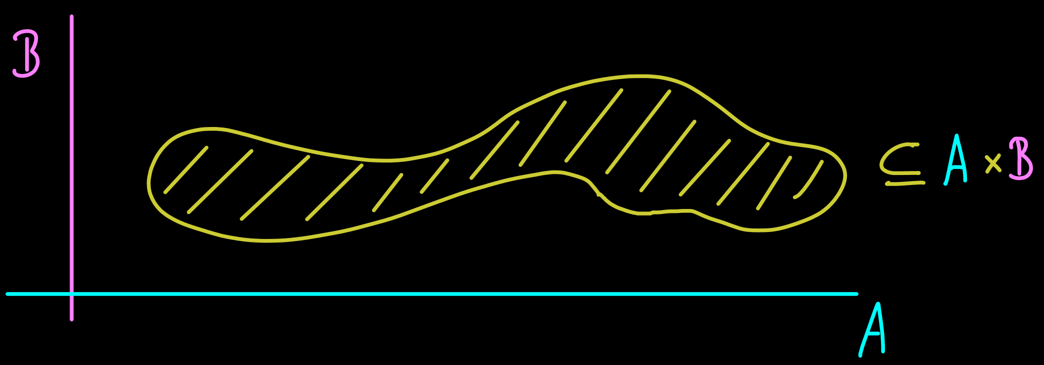
7	$(\triangle,7)$	$(\square,7)$	$(\circ,7)$
4	$(\triangle,4)$	$(\square,4)$	$(\circ,4)$
	$\triangle$	$\square$	$\circ$

Definition of ordered pair: For elements x, y write $(x,y) := \{\{x\}, \{x,y\}\}$

$$(x,y) = (\tilde{x}, \tilde{y}) \iff \{x\} = \{\tilde{x}\} \wedge \{x,y\} = \{\tilde{x}, \tilde{y}\}$$

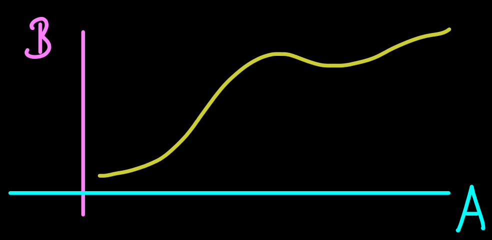
$$\iff x = \tilde{x} \wedge y = \tilde{y}$$

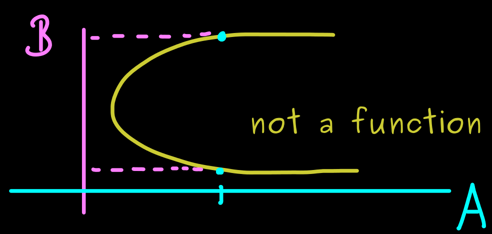
Definition: $A \times B := \{(a,b) \mid a \in A \wedge b \in B\}$



A subset $G_f \subseteq A \times B$ is called a function if

$$(\forall x\ \forall y\ \forall \tilde{y} : (x,y) \in G_f \wedge (x, \tilde{y}) \in G_f \rightarrow y = \tilde{y}) \text{ is true.}$$





If also $\forall x \in A : \exists y \in B : (x, y) \in G_f$ is true,

we write: $f : A \to B$ and $f(x) = y$ for $(x, y) \in G_f$

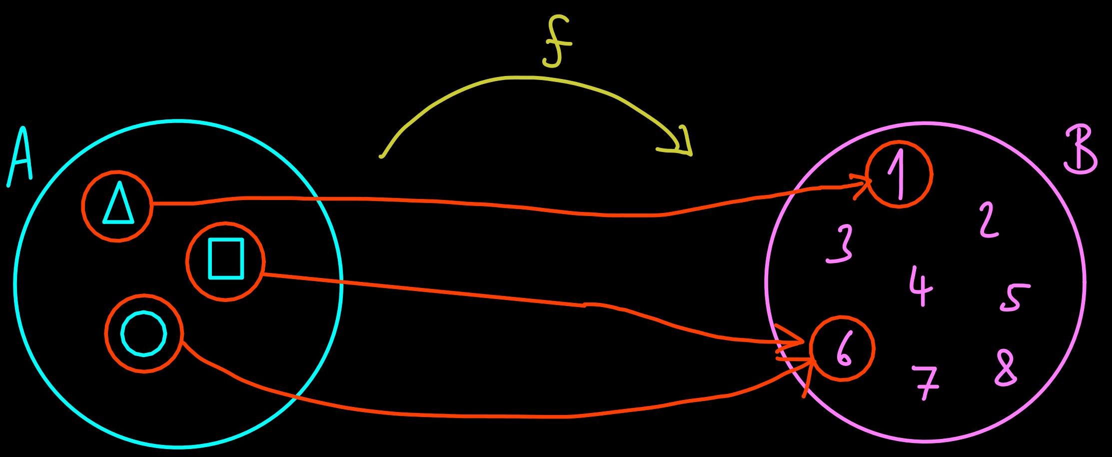
codomain of f

domain of f

a map from A into B

graph of f

Example:



$f(\Delta) = 1$

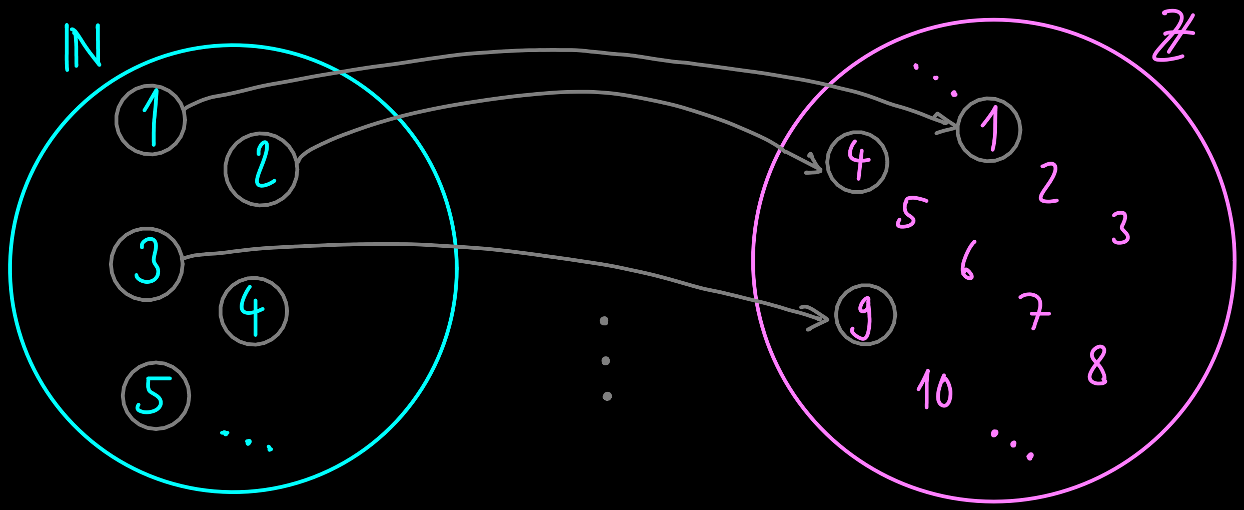
$f(\bigcirc) = 6$

$f(\square) = 6$

Start Learning Sets - Part 5

Map: $f: A \to B$

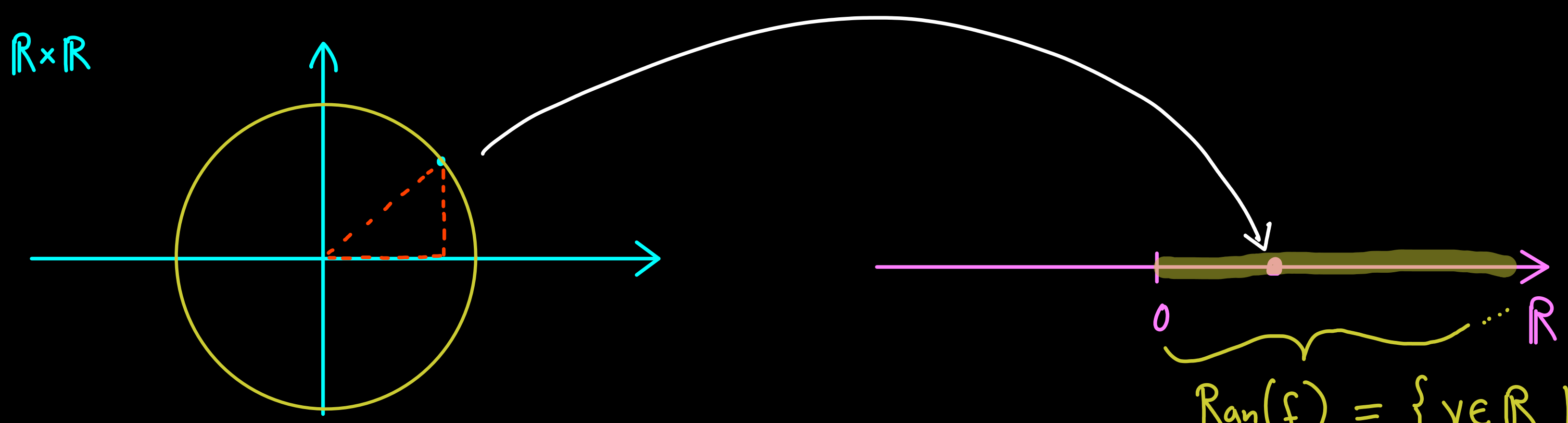
Example: $f: \mathbb{N} \to \mathbb{Z}$
$x \mapsto x^2$ new notation for $f(x) = x^2$



Range: $\operatorname{Ran}(f) := \{ y \in B \mid \exists x \in A : f(x) = y \}$

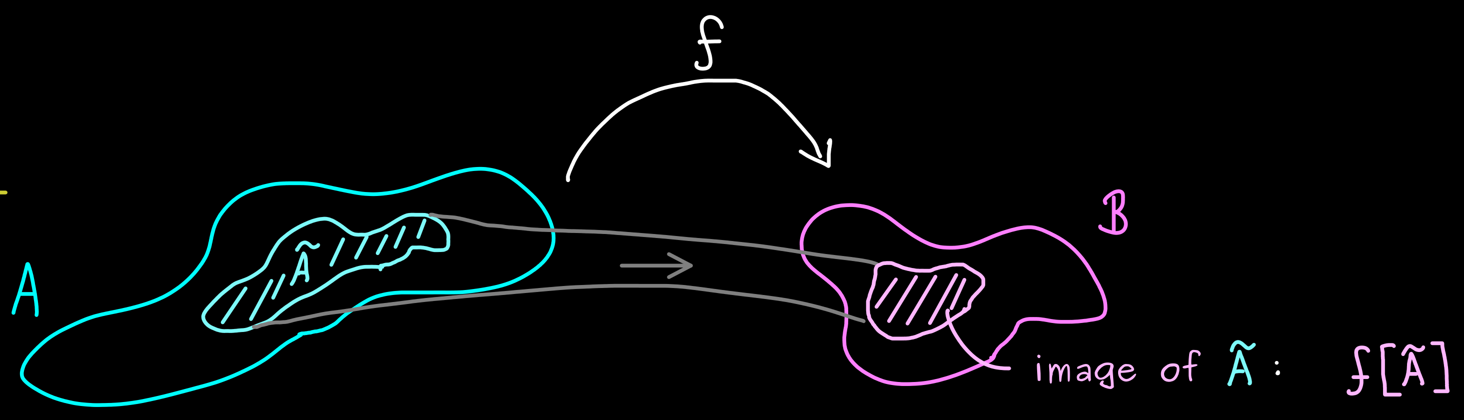
$=: \{ f(x) \mid x \in A \}$ (shorter notation)

Example: $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$
$(x_1, x_2) \mapsto x_1^2 + x_2^2$



$\operatorname{Ran}(f) = \{ y \in \mathbb{R} \mid y \geq 0 \}$

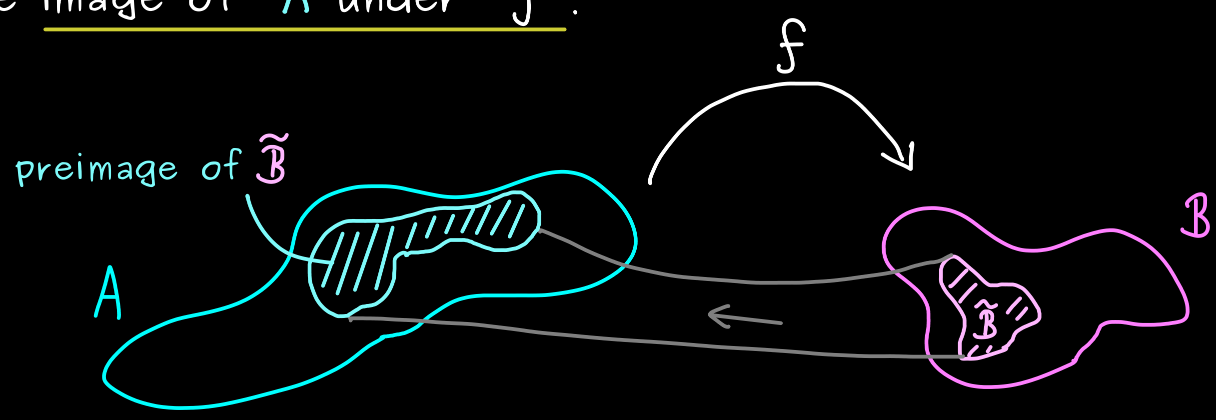
Image and preimage:



For a subset $\tilde{A} \subseteq A$,

$$f[\tilde{A}] := \{ y \in B \mid \exists x \in \tilde{A} : f(x) = y \} = \{ f(x) \mid x \in \tilde{A} \}$$

denotes the image of $\tilde{A}$ under f.



For $\tilde{B} \subseteq B$,

$$f^{-1}[\tilde{B}] := \{ x \in A \mid f(x) \in \tilde{B} \}$$

denotes the preimage of $\tilde{B}$ under f.

Example:

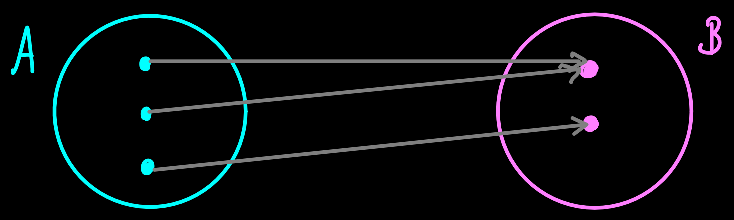
$$f : \mathbb{N} \to \mathbb{Z}$$

$$x \mapsto \begin{cases} 0 & \text{if } x \text{ even} \\ x & \text{if } x \text{ odd} \end{cases}$$

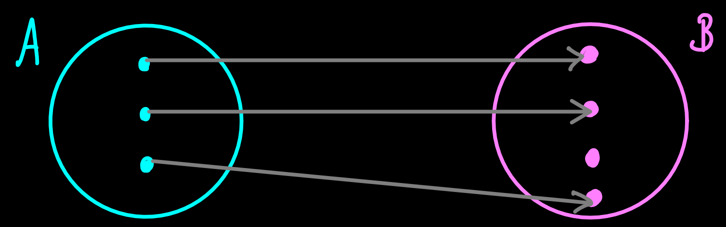
$$f[\{2,3,4\}] = \{0,3\}$$

$$f^{-1}[\{0\}] = \{2,4,6,8,10,\dots\}$$

Start Learning Sets – Part 6



not injective



not surjective

Definition: A map $f: A \to B$ is called:

injective if $\forall x_1, x_2 \in A : \left(x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2) \right)$ is true

surjective if $\forall y \in B : \exists x \in A : f(x) = y$ is true

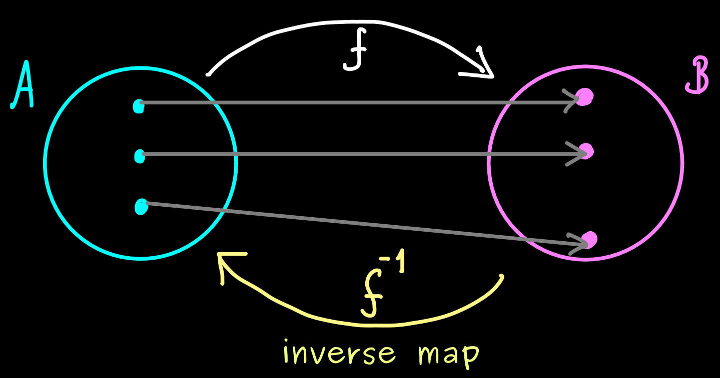
Remember:

surjective: Each $y \in B$ gets at least one arrow.

injective: Each $y \in B$ gets at most one arrow.

$\underbrace{\text{injective + surjective}}_{\text{bijective (1:1)}}$ Each $y \in B$ gets exactly one arrow.

bijective (1:1)
$\Vert$
invertible



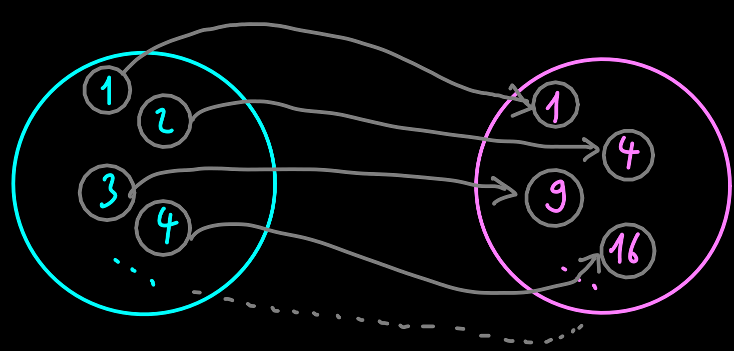
bijective

$f^{-1}: B \to A$,

$f^{-1}(y) := x$ if $f(x) = y$

Example: $f: \mathbb{N} \to \{1, 4, 9, 16, 25, 36, \ldots\}$

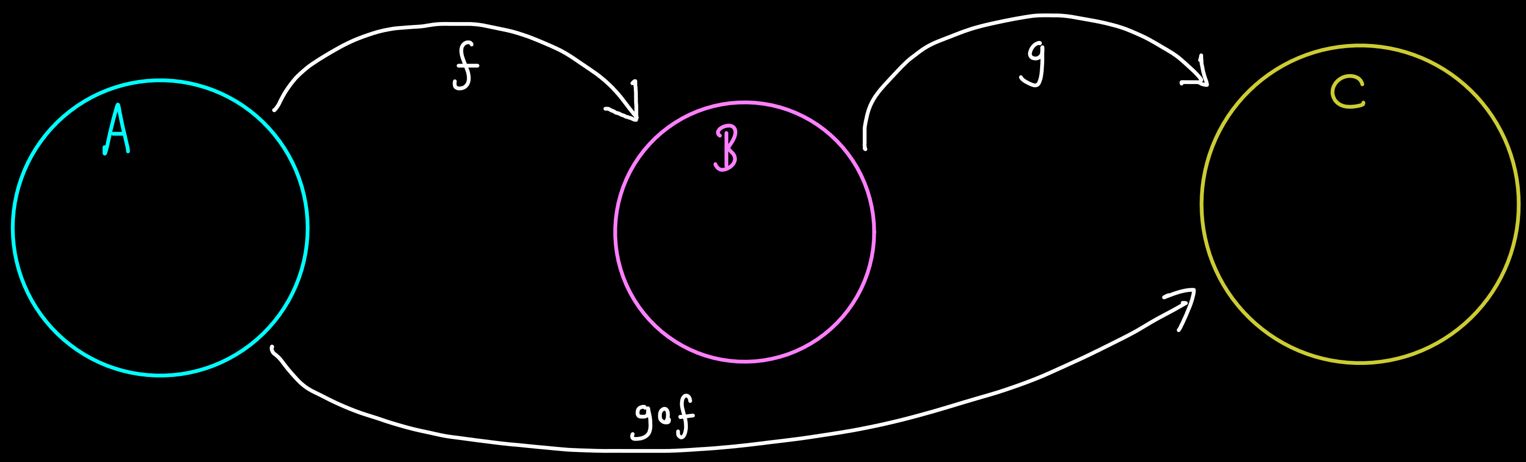
$x \mapsto x^2$



$f^{-1}: \{1, 4, 9, 16, 25, 36, \ldots\} \to \mathbb{N}$

$y \mapsto \sqrt{y}$

Start Learning Sets - Part 7

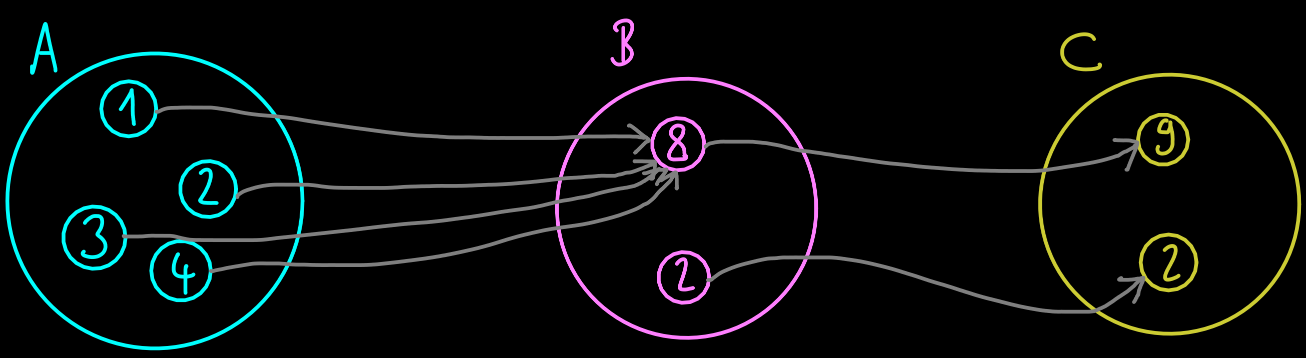


For $f: A \to B$ and $g: B \to C$ define:

$$\left.\begin{aligned} g\circ f:\ & A \to C \\ & x \mapsto g(f(x)) \end{aligned}\right\}$$ called the composition g with f

Examples:

(1)



$g(8) = 9$
$g(2) = 2$
$f(1) = 8$
$f(2) = 8$
$f(3) = 8$
$f(4) = 8$

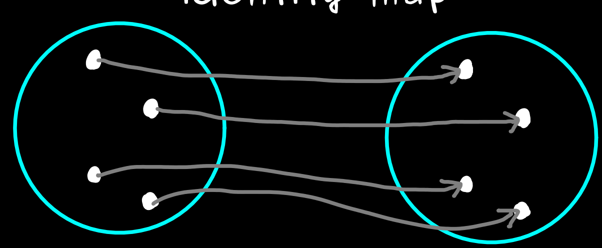
$(g\circ f)(1) = 9 \qquad (g\circ f)(2) = 9$
$(g\circ f)(3) = 9 \qquad (g\circ f)(4) = 9$

(2) $f: \mathbb{R} \to \mathbb{R}$, $x \mapsto x^2$, $g: \mathbb{R} \to \mathbb{R}$, $x \mapsto \sin(x)$

$\rightsquigarrow$ $(g\circ f)(x) = \sin(x^2)$ and $(f\circ g)(x) = (\sin(x))^2$

For any set A, we define: $id_A : A \to A$, $x \mapsto x$

identity map



For $f: A \to B$ bijective, we have:

$$f\circ f^{-1} = id_B$$
$$f^{-1}\circ f = id_A$$