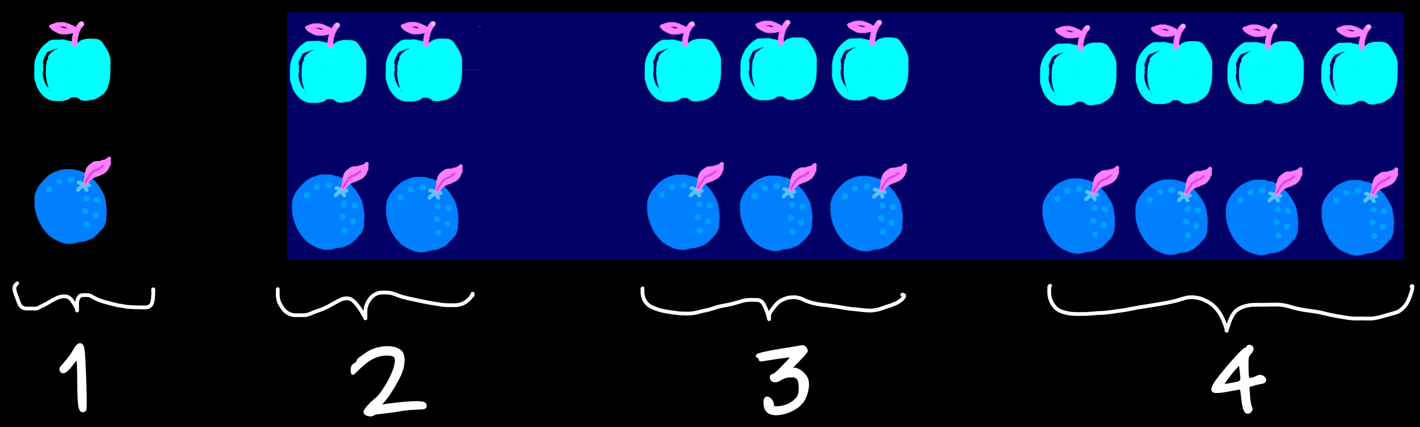
The Bright Side of Mathematics

The following pages cover the whole Start Learning Numbers course of the Bright Side of Mathematics. Please note that the creator lives from generous supporters and would be very happy about a donation. See more here: https://tbsom.de/support

Have fun learning mathematics!

Start Learning Numbers - Part 1

Natural numbers



$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

$$\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$$

$0 := \emptyset$ empty set

$1 := \{0\}$ set with one element

$2 := \{0, 1\}$ set with two elements

$3 := \{0, 1, 2\}$ set with three elements

$4 := \{0, 1, 2, 3\} = 3 \cup \{3\}$

$\vdots$

Axiom: There is a set $\mathbb{N}_0$ with the properties: (a) $0 \in \mathbb{N}_0$

(b) $\forall x: \quad x \in \mathbb{N}_0 \rightarrow x \cup \{x\} \in \mathbb{N}_0$

And $\mathbb{N}_0$ is the smallest set having these two properties.

Successor map: $s: \mathbb{N}_0 \longrightarrow \mathbb{N}_0$, $x \mapsto x \cup \{x\}$, $s(6) = 7$

Start Learning Numbers - Part 2

Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

Properties of $\mathbb{N}_0$:

(1) $0 \in \mathbb{N}_0$

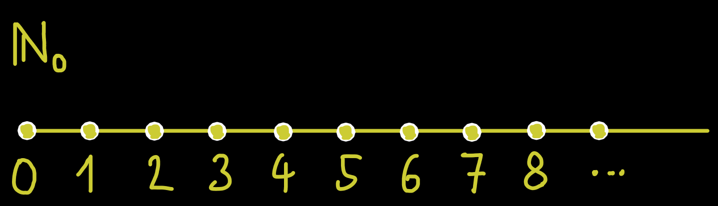
(2) There is a map $s: \mathbb{N}_0 \to \mathbb{N}_0$ that satisfies:

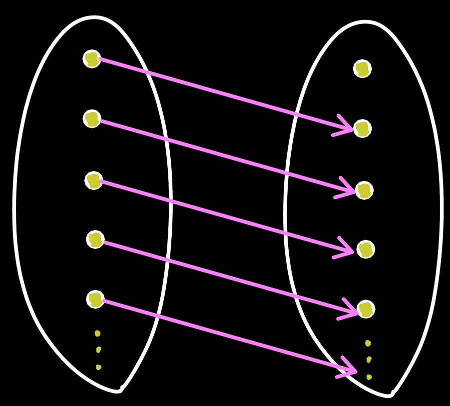
(2a) s is injective

(2b) $0 \notin \mathrm{Ran}(s) = s[\mathbb{N}_0]$

(2c) If $M \subseteq \mathbb{N}_0$ with $0 \in M$ and $s[M] \subseteq M$, then $M = \mathbb{N}_0$.

(mathematical induction)





Addition in $\mathbb{N}_0$: map

$$\mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$$
$$(m, n) \longmapsto m + n$$

How is it defined? $2 + 4 := 6$

$\boxed{m + 0 := m}$, $m + 1 := s(m)$, $m + 2 := s(m+1)$

Recursive definition: $\boxed{m + s(n) := s(m+n)}$

$$2 + 5 = 2 + s(4) = s(2+4) = s(6) = 7$$

Dedekind's principle of recursive definition:

For a set A, $a \in A$ and $h: A \to A$, then there exists a unique map $f: \mathbb{N}_0 \to A$ with $f(0) = a$ and $f(s(n)) = h(f(n))$.

(" $a, h(a), h(h(a)), h(h(h(a))), \dots$ ")

Start Learning Numbers - Part 3

Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

Each $n \in \mathbb{N}_0$ has a unique successor:

$$s: \mathbb{N}_0 \to \mathbb{N}_0 \quad , \qquad s(n) = n + 1$$

We already know: $\quad m + (n+1) = (m+n) + 1 \qquad$ (RD)

Mathematical induction:

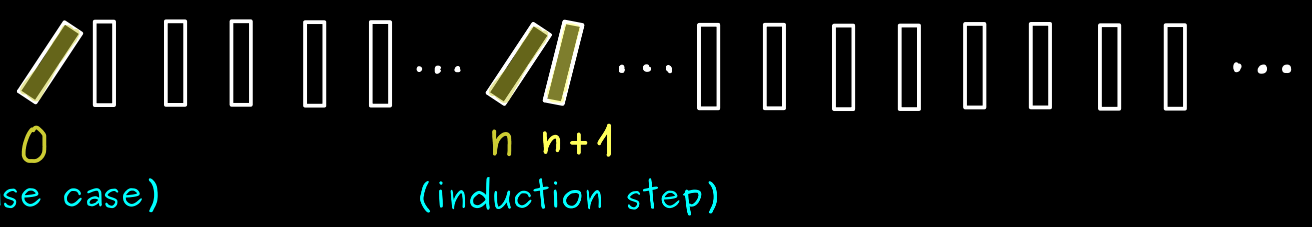
$\mathbb{N}_0$ satisfies the induction property:

Let $P(n)$ be a property for natural numbers n ("predicate").

If: (1) $P(0)$ is true (base case)

(2) $\forall n \in \mathbb{N}_0 : P(n) \to P(n+1)$ is true (induction step)

Then: $P(n)$ is true for all $n \in \mathbb{N}_0$ $\quad (\forall n : P(n)$ is true $)$



Proposition: For all $k, m, n \in \mathbb{N}_0$, we have:

$$(k+m)+n = k+(m+n) \qquad \text{(associative law)}$$

Proof: Use mathematical induction.

$P(n)$ is given by:

$$\forall k, m \in \mathbb{N}_0 : \quad (k+m)+n = k+(m+n)$$

Base case: $\mathcal{P}(0)$ means $\forall k,m \in \mathbb{N}_0 : \underbrace{(k+m)+0}_{k+m} = k + (\underbrace{m+0}_{m})$

$\Leftrightarrow \quad \forall k,m \in \mathbb{N}_0 : k+m = k+m$ true

Induction step: $(\forall n \in \mathbb{N}_0 : \mathcal{P}(n) \to \mathcal{P}(n+1))$

$$\boxed{m + (n+1) = (m+n)+1 \qquad \text{(RD)}}$$

Assume $\mathcal{P}(n)$ is true.

$\mathcal{P}(n+1)$ means $\forall k,m \in \mathbb{N}_0 : (k+m)+(n+1) = k + (m+(n+1))$

Left-hand side: $(k+m)+(n+1) \overset{\text{(RD)}}{=} ((k+m)+n)+1$

$$\overset{\mathcal{P}(n)}{=} (k+(m+n))+1$$

Right-hand side

$$\overset{\text{(RD)}}{=} k + ((m+n)+1) \overset{\text{(RD)}}{=} k + (m+(n+1))$$

Start Learning Numbers – Part 4

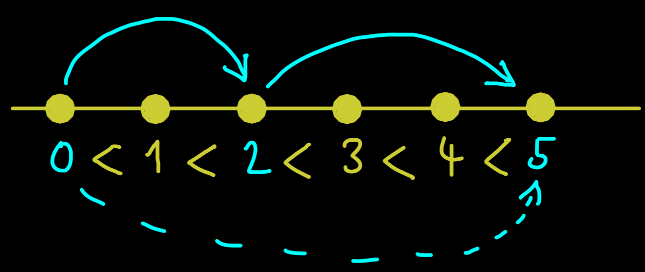
Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

Addition $+$ is a map $\mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ with:

- $m + 0 = m$ (neutral element)
- $(k + m) + n = k + (m + n)$ (associative law)
- $m + n = n + m$ (commutative law)

Ordering: We write $n \leq m$ if:

$\exists k \in \mathbb{N}_0 : \quad m = n + k$



And we write $n < m$ if: $\quad n \leq m \quad \wedge \quad n \neq m$

Properties:

(1) $n \leq n$ (reflexive)

(2) If $n \leq m \ \wedge \ m \leq n$, then $n = m$ (antisymmetric)

(3) If $n \leq \ell \ \wedge \ \ell \leq m$, then $n \leq m$ (transitive)

Proof: Assume $n \leq \ell$ and $\ell \leq m$ are true. So:

$\exists k_1 \in \mathbb{N}_0 : \ \ell = n + k_1$ and $\exists k_2 \in \mathbb{N}_0 : \ m = \ell + k_2$ are true.

Therefore: $\quad m = \ell + k_2 = (n + k_1) + k_2$

$$= n + \underbrace{(k_1 + k_2)}_{=: k \in \mathbb{N}_0} = n + k$$

Therefore: $\exists k \in \mathbb{N}_0 : \ m = n + k$ is true, so $n \leq m$ is true.

Start Learning Numbers - Part 5

Natural numbers: $\mathbb{N}_0 = \{0, 1, 2, 3, 4, \dots\}$

$$\underbrace{4 + 4 + 4 + 4 + 4}_{\text{We have } 5 \text{ of them}} =: 5 \cdot 4$$

$$3 + 3 + 3 + 3 + 3 + 3 =: 6 \cdot 3$$

$$4 =: 1 \cdot 4$$

$$0 =: 0 \cdot 4$$

How can we define the multiplication?

Multiplication in $\mathbb{N}_0$: map $\mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$

$(n, m) \longmapsto n \cdot m$ defined by

$$0 \cdot m := 0$$
$$(n+1) \cdot m := (n \cdot m) + m$$

(recursive definition)

$$5 \cdot 2 = 2 + 2 + 2 + 2 + 2$$
$$6 \cdot 2 = \underbrace{2 + 2 + 2 + 2 + 2}_{5 \cdot 2} + 2$$

(Map is well-defined by Dedekind's recursion theorem)

Properties:

(1) $n \cdot (m \cdot k) = (n \cdot m) \cdot k$ (associative)

(2) $n \cdot m = m \cdot n$ (commutative)

(3) $1 \cdot m = m$ (neutral element)

How to connect $+$ and $\cdot$: $n \cdot (m+k) = n \cdot m + n \cdot k$ (distributive)

$$\boxed{\begin{aligned} 0 \cdot m &:= 0 \\ (n+1) \cdot m &:= (n \cdot m) + m \end{aligned}} \quad (*)$$

Proof by induction: Base case: $n = 0$

Left-hand side: $0 \cdot (m+k) = 0$ ✓

Right-hand side: $0 \cdot m + 0 \cdot k = 0 + 0 = 0$ ✓

Induction step: Assume $n \cdot (m+k) = n \cdot m + n \cdot k$ holds for n.
(induction hypothesis)

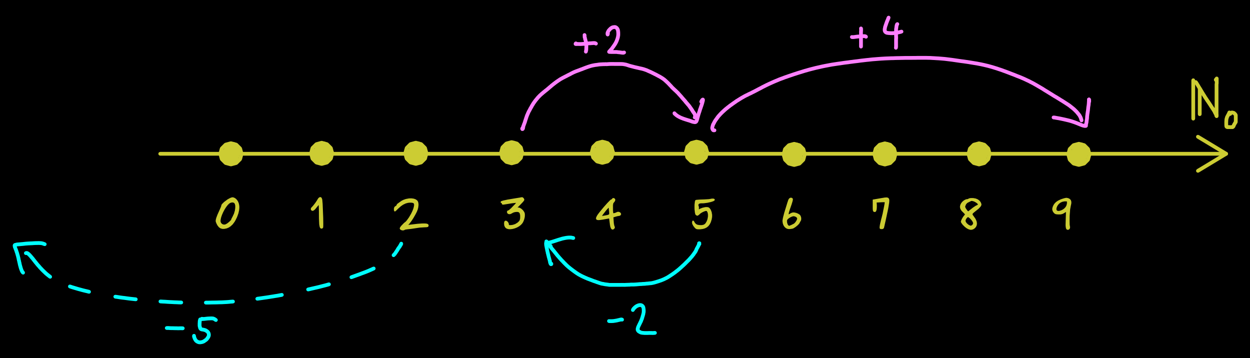
Left-hand side: $(n+1) \cdot (m+k) \overset{(*)}{=} n \cdot (m+k) + (m+k)$

$$\overset{(i.h.)}{=} n \cdot m + (n \cdot k + m) + k$$

$$= (n \cdot m + m) + (n \cdot k + k)$$

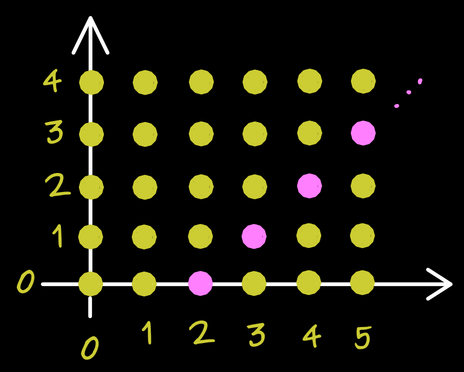
$$\overset{(*)}{=} (n+1) \cdot m + (n+1) \cdot k \quad \leftarrow \text{Right-hand side}$$

Start Learning Numbers — Part 6



$5-2=3$ $\qquad$ $2-5=?$

Idea: Look at pairs $(9,5)$, $(5,3)$, $(3,5)$



$\vdots$
$(5,3)$
$(4,2)$
$(3,1)$
$(2,0)$

should be the "same"

$$\mathbb{N}_0 \times \mathbb{N}_0 =: \mathbb{N}_0^2$$

$(5,3)$ stands for "$5-3$"
$\qquad\qquad\qquad\qquad$ "
$(4,2)$ stands for "$4-2$"

$5-3 = 4-2$ ← not okay

$5+2 = 4+3$ ← totally okay

$(5,3) \sim (4,2)$ (equivalent)

Equivalence relation: We write $(a,b) \sim (x,y)$ if:

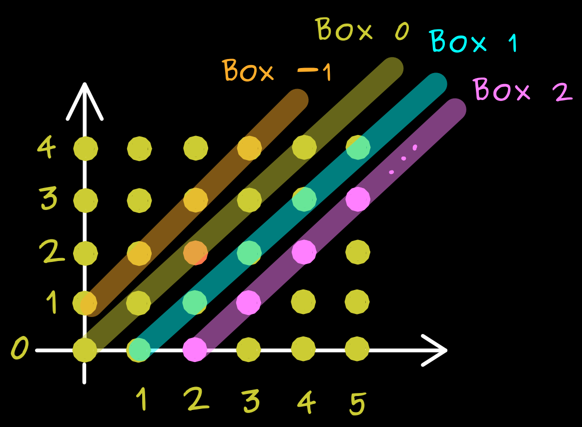
$$a+y = x+b$$

Properties:

(1) $(a,b) \sim (a,b)$ (reflexive)

(2) If $(a,b) \sim (x,y)$, then $(x,y) \sim (a,b)$. (symmetric)

(3) If $(a,b) \sim (x,y)$ and $(x,y) \sim (c,d)$, then $(a,b) \sim (c,d)$. (transitive)



Property of $\mathbb{N}_0$ (cancellation):

If $m + n = \tilde{m} + n$, then $m = \tilde{m}$.

$$\text{Box } 0 = [(2,2)]_{\sim} := \{(x,y) \in \mathbb{N}_0^2 \mid (x,y) \sim (2,2)\}$$

is called the equivalence class of $(2,2)$.

$\text{Box } 0 = [(0,0)]_{\sim} = [(2,2)]_{\sim}$

$\text{Box } 1 = [(1,0)]_{\sim} = [(9,8)]_{\sim}$

$\text{Box } 2 = [(2,0)]_{\sim}$

$\vdots \qquad \vdots$

$\text{Box } {-1} = [(0,1)]_{\sim} = [(8,9)]_{\sim}$

$\text{Box } {-2} = [(0,2)]_{\sim}$

$\vdots \qquad \vdots$

$\mathbb{Z} :=$ set of all boxes (equivalence classes)

Start Learning Numbers - Part 7

In $\mathbb{N}_0$ $\quad 4 + x = 0 \quad$ is not solvable! No "inverse" of 4.

$$\mathbb{Z} := \left\{ [(a,b)]_\sim \;\middle|\; (a,b) \in \mathbb{N}_0^2 \right\} =: \mathbb{N}_0^2 / \sim$$

with $[(a,b)]_\sim := \left\{ (x,y) \;\middle|\; (x,y) \sim (a,b) \right\}$

and $(x,y) \sim (a,b) \;\Leftrightarrow\; x + b = a + y$

$[(0,0)]_\sim =: 0_\mathbb{Z}$
$[(1,0)]_\sim =: 1_\mathbb{Z}$
$[(2,0)]_\sim =: 2_\mathbb{Z}$
$\vdots$

$[(0,1)]_\sim =: (-1)_\mathbb{Z}$
$[(0,2)]_\sim =: (-2)_\mathbb{Z}$
$\vdots$

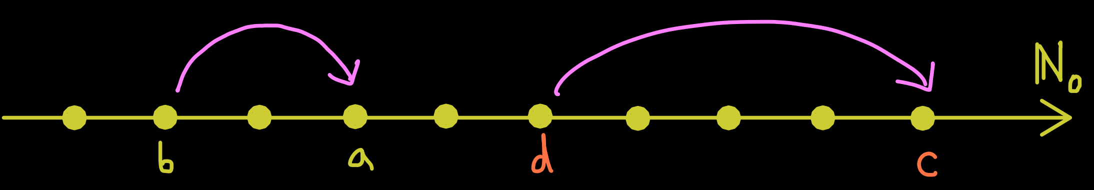
$$\mathbb{Z} = \left\{ \ldots, (-2)_\mathbb{Z}, (-1)_\mathbb{Z}, 0_\mathbb{Z}, 1_\mathbb{Z}, 2_\mathbb{Z}, \ldots \right\}$$

Question: Is $4_\mathbb{Z} + x = 0_\mathbb{Z}$ now solvable? And with $x = (-4)_\mathbb{Z}$?

First question: How is $+$ as a map $\mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ defined?

$$[(a,b)]_\sim + [(c,d)]_\sim := [(a+c, b+d)]_\sim$$

well-defined? ✓



Take $(\tilde{a},\tilde{b}) \sim (a,b)$ and $(\tilde{c},\tilde{d}) \sim (c,d)$. Then $[(\tilde{a},\tilde{b})]_\sim + [(\tilde{c},\tilde{d})]_\sim = [(\tilde{a}+\tilde{c}, \tilde{b}+\tilde{d})]_\sim$

Is $(\tilde{a}+\tilde{c}, \tilde{b}+\tilde{d}) \sim (a+c, b+d)$?

Proof:
$$\left.\begin{array}{l} (\tilde{a},\tilde{b}) \sim (a,b) \iff \tilde{a}+b = a+\tilde{b} \\ (\tilde{c},\tilde{d}) \sim (c,d) \iff \tilde{c}+d = c+\tilde{d} \end{array}\right\} \text{implies: } \tilde{a}+\tilde{c}+b+d = a+c+\tilde{b}+\tilde{d}$$
$$\iff (\tilde{a}+\tilde{c},\, \tilde{b}+\tilde{d}) \sim (a+c,\, b+d)$$

Examples: (a) $4_{\mathbb{Z}} + 2_{\mathbb{Z}} = [(4,0)]_{\sim} + [(2,0)]_{\sim} = [(6,0)]_{\sim} = 6_{\mathbb{Z}}$

(b) $4_{\mathbb{Z}} + (-4)_{\mathbb{Z}} = [(4,0)]_{\sim} + [(0,4)]_{\sim} = [(4,4)]_{\sim} = [(0,0)]_{\sim} = 0_{\mathbb{Z}}$

Properties of $\mathbb{Z}$ together with $+$: map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

(a) associative

(b) commutative

(c) $m + 0_{\mathbb{Z}} = m$ ($0_{\mathbb{Z}}$ is neutral element)

(d) For all $m \in \mathbb{Z}$, there is an element $\tilde{m} \in \mathbb{Z}$ with $m + \tilde{m} = 0_{\mathbb{Z}}$

$\to$ $(\mathbb{Z}, +)$ is an abelian group

Start Learning Numbers – Part 8

$$\mathbb{Z} = \{ \ldots, (-2)_{\mathbb{Z}}, (-1)_{\mathbb{Z}}, 0_{\mathbb{Z}}, 1_{\mathbb{Z}}, 2_{\mathbb{Z}}, \ldots \}$$

$2_{\mathbb{Z}} = [(6,4)]_{\sim}$ ← think of "6−4"

think of "$(a-b)\cdot(c-d) = (ac+bd)-(ad+bc)$"

$$[(a,b)]_{\sim} \cdot [(c,d)]_{\sim} := [(a\cdot c + b\cdot d,\ a\cdot d + b\cdot c)]_{\sim}$$

The multiplication is well-defined.

Properties of $\mathbb{Z}$ together with $\cdot$:

← map $\mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$

(a) associative

(b) commutative

(c) $1_{\mathbb{Z}} \cdot m = m$ ($1_{\mathbb{Z}}$ is neutral element)

(d) distributive

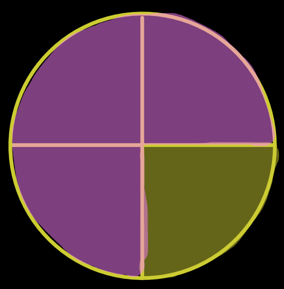
Examples:

(a) $4_{\mathbb{Z}} \cdot 2_{\mathbb{Z}} = [(4,0)]_{\sim} \cdot [(2,0)]_{\sim} = [(4\cdot 2 + 0\cdot 0,\ 4\cdot 0 + 0\cdot 2)]_{\sim} = 8_{\mathbb{Z}}$

(b) $(-4)_{\mathbb{Z}} \cdot (-2)_{\mathbb{Z}} = [(0,4)]_{\sim} \cdot [(0,2)]_{\sim} = [(0\cdot 0 + 4\cdot 2,\ 0\cdot 2 + 4\cdot 0)]_{\sim} = 8_{\mathbb{Z}}$

Start Learning Numbers – Part 9

$$\mathbb{Z} = \{\dots -3, -2, -1, 0, 1, 2, 3, 4, \dots\}$$



ratio: $3:1$ or $3:4$ or $1:4$

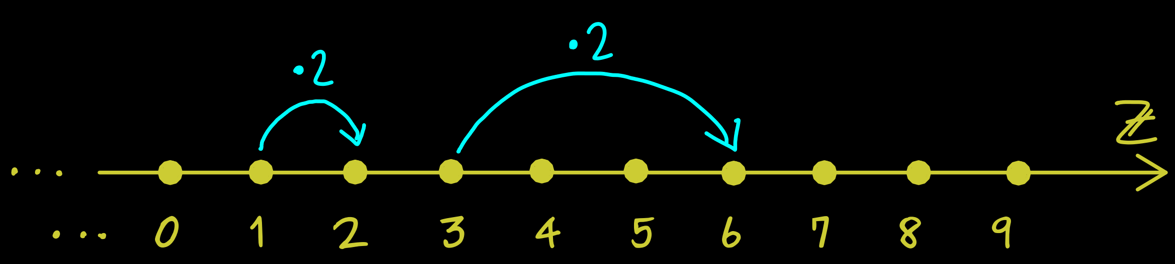
fraction: $\frac{3}{4} + \frac{1}{4} = 1$

solve $4 \cdot x = 1$? $\rightsquigarrow$ We need inverses with respect to $\cdot$!

Works the same as $(\mathbb{N}_0, +) \rightsquigarrow (\mathbb{Z}, +)$

For $(c, d), (a, b) \in \mathbb{Z} \times \mathbb{Z}\setminus\{0\}$ define:

$$(a, b) \sim (c, d) \quad \text{by} \quad a \cdot d = c \cdot b$$



$(6,3) \sim (2,1)$ "$\frac{6}{3} = \frac{2}{1}$"

$$\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z}\setminus\{0\})/\sim \; = \left\{ [(a,b)]_\sim \;\middle|\; (a,b) \in \mathbb{Z} \times \mathbb{Z}\setminus\{0\} \right\}$$ rational numbers

Examples: $[(4,2)]_\sim = [(6,3)]_\sim = [(2,1)]_\sim =: 2_{\mathbb{Q}}$

$[(-9,-3)]_\sim = [(9,3)]_\sim = [(3,1)]_\sim =: 3_{\mathbb{Q}}$, $[(0,8)]_\sim = [(0,1)]_\sim =: 0_{\mathbb{Q}}$

$[(-9,3)]_\sim = [(-3,1)]_\sim =: (-3)_{\mathbb{Q}}$ We get all integers back!

$[(2,8)]_\sim = [(1,4)]_\sim =: \left(\frac{1}{4}\right)_{\mathbb{Q}}$ $\rightsquigarrow$ fractions

Definition: $[(a,b)]_\sim =: \frac{a}{b}$ $\left(\frac{2}{8} = \frac{1}{4} \right)$

Start Learning Numbers - Part 10

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z},\ b \in \mathbb{Z}\setminus\{0\} \right\}, \qquad \frac{a}{b} = \frac{c}{d} \;\Leftrightarrow\; a\cdot d = c\cdot b$$

Multiplication: $\frac{a}{b} \cdot \frac{c}{d} := \frac{a\cdot c}{b\cdot d}$ well-defined!

For $a \neq 0$, we have: $\frac{a}{b} \cdot \frac{b}{a} = \frac{a\cdot b}{b\cdot a} = \frac{1}{1} \; (= 1_{\mathbb{Q}})$

Solve: $4 \cdot x = 1$? In $\mathbb{Q}$: $\frac{4}{1} \cdot x = \frac{1}{1}$ is solved by: $x = \frac{1}{4}$

Property: $(\mathbb{Q}\setminus\{0_{\mathbb{Q}}\}, \cdot)$ is an abelian group.

How to define the addition?

We want the distributive law:

$$\boxed{\frac{a}{d} + \frac{c}{d}} = \frac{a}{1}\cdot\frac{1}{d} + \frac{c}{1}\cdot\frac{1}{d} = \left(\frac{a}{1} + \frac{c}{1}\right)\cdot\frac{1}{d} = \boxed{\frac{a+c}{d}}$$

should be defined by:

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b}\cdot\frac{d}{d} + \frac{c}{d}\cdot\frac{b}{b} = \frac{a\cdot d}{1}\cdot\frac{1}{b\cdot d} + \frac{c\cdot b}{1}\cdot\frac{1}{b\cdot d}$$

$$= \left(\frac{a\cdot d}{1} + \frac{c\cdot b}{1}\right)\cdot\frac{1}{b\cdot d} = \frac{a\cdot d + c\cdot b}{b\cdot d}$$

Define: $\frac{a}{b} + \frac{c}{d} := \frac{a\cdot d + c\cdot b}{b\cdot d}$ well-defined!

Proposition: The set $\mathbb{Q}$ together with the operation $+$ and $\cdot$ satifies:

(1) $(\mathbb{Q}, +)$ is an abelian group

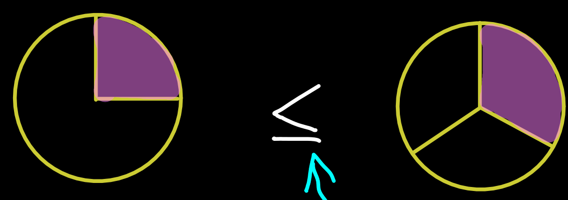
(2) $(\mathbb{Q}\setminus\{0_{\mathbb{Q}}\}, \cdot)$ is an abelian group

(3) distributive law

field

Start Learning Numbers - Part 11

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z},\ b \in \mathbb{Z}\setminus\{0\} \right\}$$



$\leq$ We need to define it

Definition of $\leq$ for $\mathbb{Z}$: For $a, b \in \mathbb{Z}$, we write $a \leq b$ if $\exists k \in \mathbb{N}_0 : a + k = b$

Now: $\frac{1}{4} \leq \frac{1}{3}$ because $3 \leq 4$

Definition of $\leq$ for $\mathbb{Q}$: For $b > 0$ and $d > 0$

$$\frac{a}{b} \leq \frac{c}{d} \quad \text{defined by} \quad a \cdot d \leq c \cdot b$$

Properties of $\leq$ for $\mathbb{Q}$:

(1) Ordering: reflexive, antisymmetric and transitive.

(2) For all $x, y, z \in \mathbb{Q}$: If $x \leq y$, then $x + z \leq y + z$

(3) For all $x, y, z \in \mathbb{Q}$: If $z \geq 0$ and $x \leq y$, then $x \cdot z \leq y \cdot z$

(4) Total order: For all $x, y \in \mathbb{Q}$, we have $x \leq y$ or $y \leq x$.

(5) Archimedean property: For all $x, \varepsilon \in \mathbb{Q}$ with $x > 0$ and $\varepsilon > 0$,

we have: $n \in \mathbb{N}_0 : n \cdot \varepsilon = \varepsilon + \varepsilon + \varepsilon + \cdots + \varepsilon > x$