The Bright Side of
Mathematics

Start Learning Numbers – Part 10

$$\mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a \in \mathbb{Z},\ b \in \mathbb{Z}\setminus\{0\} \right\}, \quad \frac{a}{b} = \frac{c}{d} \iff a \cdot d = c \cdot b$$

Multiplication: $\frac{a}{b} \cdot \frac{c}{d} := \frac{a \cdot c}{b \cdot d}$ well-defined!

For $a \neq 0$, we have: $\frac{a}{b} \cdot \frac{b}{a} = \frac{a \cdot b}{b \cdot a} = \frac{1}{1} \left(= 1_{\mathbb{Q}} \right)$

Solve: $4 \cdot x = 1$? In $\mathbb{Q}$: $\frac{4}{1} \cdot x = \frac{1}{1}$ is solved by: $x = \frac{1}{4}$

Property: $\left(\mathbb{Q}\setminus\{0_{\mathbb{Q}}\}, \cdot\right)$ is an abelian group.

How to define the addition?

We want the distributive law:

$$\frac{a}{d} + \frac{c}{d} = \frac{a}{1} \cdot \frac{1}{d} + \frac{c}{1} \cdot \frac{1}{d} = \left(\frac{a}{1} + \frac{c}{1} \right) \cdot \frac{1}{d} = \frac{a + c}{d}$$

should be defined by:

$$\frac{a}{b} + \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{d} + \frac{c}{d} \cdot \frac{b}{b} = \frac{a \cdot d}{1} \cdot \frac{1}{b \cdot d} + \frac{c \cdot b}{1} \cdot \frac{1}{b \cdot d}$$

$$= \left(\frac{a \cdot d}{1} + \frac{c \cdot b}{1} \right) \cdot \frac{1}{b \cdot d} = \frac{a \cdot d + c \cdot b}{b \cdot d}$$

Define: $\frac{a}{b} + \frac{c}{d} := \frac{a \cdot d + c \cdot b}{b \cdot d}$ well-defined!

Proposition: The set $\mathbb{Q}$ together with the operation $+$ and $\cdot$ satifies:

(1) $(\mathbb{Q}, +)$ is an abelian group

(2) $\left(\mathbb{Q}\setminus\{0_{\mathbb{Q}}\}, \cdot\right)$ is an abelian group

(3) distributive law

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